Semi-tensor Product of Matrices:Concepts and Properties

Series One, Lesson One

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Outline

I. What is STP

☞ Matrix-Matrix Left Semi-tensor Product (MM-L STP)

Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$. The Kronecker product of A and B is

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}. \tag{1}$$

Definition 1.1

Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, $t = lcm(n,p)$. Then the (MM-L) STP of A and B is

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right). \tag{2}$$

Example 1.2

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}.$$

Then $t = lcm(3,2) = 6$, and

$$A \ltimes B = (A \otimes I_2)(B \times I_3)$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 2 & 0 \\ 0 & -1 & 0 & 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{bmatrix}$$

Example 1.2(cont'd)

$$= \begin{bmatrix} -1 & -1 & 0 & 0 & -3 & 0 \\ 0 & -1 & -1 & 0 & 0 & -3 \\ -2 & 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 3 & 0 & 0 \\ 1 & 2 & -3 & 0 & 6 & 0 \\ 3 & 1 & 2 & 9 & 0 & 6 \end{bmatrix}$$

Remark 1.3

(i) Since it is a product between two matrices, it is called "MM". "MV" will be defined later.

(ii) "Left" STP corresponds to "Right" STP, which is defined right after this Remark.

(iii) The MM-L is the fundamental one, hence it is considered as the default STP.

MM-R, MV-L, MV-R STPs

Definition 1.4

Let $A \in \mathcal{M}_{m\times n},\ B \in \mathcal{M}_{p\times q}, x \in \mathbb{R}^p,\ t = lcm(n,p)$.

(i) The (MM-R) STP of A and B is

$$A \rtimes B := \left(I_{t/n} \otimes A\right)\left(I_{t/p} \otimes B\right). \qquad (3)$$

(ii) The (MV-L) STP of A and x is

$$A \vec{\ltimes} x := \left(A \otimes I_{t/n}\right)\left(x \otimes \mathbf{1}_{t/p}\right). \qquad (4)$$

(iii) The (MV-R) STP of A and x is

$$A \vec{\rtimes} x := \left(I_{t/n} \otimes A\right)\left(\mathbf{1}_{t/p} \otimes x\right). \qquad (5)$$

II. Properties of STP

☞ Basic Properties

Proposition 2.1

(In the following: $\bowtie \in \{\ltimes, \rtimes\}$.)

- (Assosiativity)

$$(A \bowtie B) \bowtie C = A \bowtie (B \bowtie C). \tag{6}$$

- (Distributivity)

$$\begin{aligned} (A + B) \bowtie C &:= A \bowtie C + B \bowtie C \\ A \bowtie (B + C) &= A \bowtie B + A \bowtie C. \end{aligned} \tag{7}$$

Proposition 2.1 (cont'd)

- (Transpose)

$$(A \bowtie B)^T = B^T \bowtie A^T. \tag{8}$$

- (Inverse)
  If A and B are invertible, then $A \bowtie B$ is invertible. Moreover,

$$(A \bowtie B)^{-1} = B^{-1} \bowtie A^{-1}. \tag{9}$$

Assume $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, then we say

$$A : B := n : p.$$

The STP of A and B depends only on $A : B$.

Definition 2.2

Assume $A : B = n : p$. A splitting

$$A = \begin{bmatrix} A^{11} & A^{12} & \cdots & A^{1\ell} \\ A^{21} & A^{22} & \cdots & A^{2\ell} \\ \vdots & & & \\ A^{s1} & A^{s2} & \cdots & A^{s\ell} \end{bmatrix}, \quad B = \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1t} \\ B^{21} & B^{22} & \cdots & A^{2t} \\ \vdots & & & \\ B^{\ell 1} & B^{\ell 2} & \cdots & B^{\ell t} \end{bmatrix} \tag{10}$$

is called a proper division, if

$$A^{i\alpha} : B^{\alpha j} = n : p, \quad i = 1, 2, \cdots, s;\ j = 1, 2, \cdots, t.$$

Theorem 2.3

Assume $A : B = n : p$, and the splitting (10) is a proper division, then

$$A \ltimes B = \left(C^{ij}\right), \tag{11}$$

where

$$C^{ij} = \sum_{k=1}^{\ell} A^{ik} \ltimes B^{kj}.$$

Corollary 2.4

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Then

$$A \ltimes B := \left(C^{i,j} \mid i = 1, \cdots, m;\ j = 1, \cdots, q\right), \tag{12}$$

where

$$C^{ij} = \mathrm{Row}_i(A) \ltimes \mathrm{Col}_j(B).$$

Remark 2.5

Theorem 2.3 and Corollary 2.4 are not correct for MM-R STP. This is the major difference between MM-L SPT and MM-R STP. Mainly because of this difference, MM-R is not so useful as MM-L.

Proposition 2.6

Let $A,\ B \in \mathcal{M}_{m\times n}$, where $n = \prod_{i=1}^{n} n_i$. If

$$A \ltimes_{i=1}^{n} X_i = B \ltimes_{i=1}^{n} X_i, \quad \forall X_i \in \Delta_{n_i},\ i = 1, \cdots, n, \tag{13}$$

then

$$A = B.$$

☞ STP vs Kronecker Product

Proposition 2.7

(i) Given two column vectors $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$, then

$$X \ltimes Y = X \otimes Y; \tag{14}$$

$$X \rtimes Y = Y \otimes X. \tag{15}$$

(ii) Given two row vectors $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$, then

$$\xi \ltimes \eta = \eta \otimes \xi; \tag{16}$$

$$\xi \rtimes \eta = \xi \otimes \eta. \tag{17}$$

Proposition 2.8

Assume $x = \ltimes_{i=1}^{n} x_i$, where $x_i \in \Delta_{n_i}$, $i = 1, 2, \cdots, n$. Define

$$\begin{aligned} p^t &:= \begin{cases} 1, & t = 1, \\ \prod_{i=1}^{t-1} n_i, & t = 2, 3, \cdots, n, \end{cases} \\ q^t &:= \begin{cases} 1, & t = n, \\ \prod_{i=t+1}^{n} n_i, & t = 1, 2, 3, \cdots, n-1. \end{cases} \end{aligned}$$

Then for any $1 \le j \le n$ we have

$$x_j = \left[\mathbf{1}_{p^j}^T \otimes I_{n_j} \otimes \mathbf{1}_{q^T}^T \right] x, \quad j = 1, 2, \cdots, n, \tag{18}$$

☞ Pseudo-Commutativity

Proposition 2.9

(i) Assume $X \in \mathbb{R}^t$ is a column vector, A is a matrix, then

$$XA = (I_t \otimes A)X. \tag{19}$$

(ii) Assume $\omega \in \mathbb{R}^t$ is a row vector, A is a matrix, then

$$A\omega = \omega(I_t \otimes A). \tag{20}$$

Definition 2.10

A swap matrix of dimension (m,n)-is defined as follows:

$$W_{[m,n]} := \left[I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \cdots, I_n \otimes \delta_m^m\right]. \tag{21}$$

Proposition 2.11

(i)

$$W_{[m,n]}^T := W_{[n,m]}. \tag{22}$$

(ii)

$$W_{[m,n]}^{-1} := W_{[m,n]}^T. \tag{23}$$

Proposition 2.12

(i) Let $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$ be two column vectors. Then

$$W_{[m,n]} X \ltimes Y = Y \ltimes X. \tag{24}$$

(ii) Let $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$ be two row vectors. Then

$$\xi \ltimes \eta W_{[m,n]} = \eta \ltimes \xi. \tag{25}$$

Proposition 2.13

Let $A \in M_{m \times n}$. Then

$$\begin{cases} W_{[m,n]} V_r(A) = V_c(A), \\ W_{[n,m]} V_c(A) = V_r(A). \end{cases} \tag{26}$$

Proposition 2.14

The swap matrix $W_{[m,n]}$ has two equivalent forms:

(i)

$$W_{[m,n]} = \left[\delta_n^1 \ltimes \delta_m^1 \quad \cdots \quad \delta_n^n \ltimes \delta_m^1 \quad \cdots \quad \delta_n^1 \ltimes \delta_m^m \quad \cdots \quad \delta_n^n \ltimes \delta_m^m\right]. \tag{27}$$

(ii)

$$W_{[m,n]} = \begin{bmatrix} I_m \otimes {\delta_n^1}^T \\ I_m \otimes {\delta_n^2}^T \\ \vdots \\ I_m \otimes {\delta_n^n}^T \end{bmatrix}. \tag{28}$$

Proposition 2.15

Let $x_i \in \mathbb{R}^{n_i}$, $i = 1, \cdots, n$. Denote $p = \prod_{i=1}^{j-1} n_i$ $q = \prod_{i=j+2}^{n} n_i$, and define

$$I_p \otimes W_{[d_j, d_{j+1}]} \otimes I_q. \tag{29}$$

Then we have

$$\left(I_p \otimes W_{[d_j, d_{j+1}]} \otimes I_q\right) \ltimes_{i=1}^{n} x_i = x_1 x_2 \cdots x_{j+1} x_j \cdots x_n. \tag{30}$$

Proposition 2.16

$$\begin{aligned} W_{[p,qr]} &= (I_q \otimes W_{[p,r]})(W_{[p,q]} \otimes I_r), \\ W_{[p,qr]} &= (I_r \otimes W_{[p,q]})(W_{[p,r]} \otimes I_q). \end{aligned} \tag{31}$$

$$\begin{aligned} W_{[pq,r]} &= (W_{[p,r]} \otimes I_q)(I_p \otimes W_{[q,r]}), \\ W_{[pq,r]} &= (W_{[q,r]} \otimes I_p)(I_q \otimes W_{[p,r]}). \end{aligned} \tag{32}$$

Proposition 2.17

Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$. Then

$$W_{[m,p]}(A \otimes B)W_{[q,n]} = B \otimes A. \tag{33}$$

☞ What is the basic idea inside STP

Remark 2.18

The basic idea:

Mismatching Dim's $\Rightarrow$ Enlarging Factors $\Rightarrow$ Matching Dim's.

☞ Cross-dimensional Vector Space

Example 2.19

$$\mathbb{R}^{\sigma} = \bigcup_{n=1}^{\infty} \mathbb{R}^{n}.$$

Example 2.19(cont'd)

$X \in \mathbb{R}^p \subset \mathbb{R}^\sigma, \qquad Y \in \mathbb{R}^q \subset \mathbb{R}^\sigma.$

(i) $p = q$

$$\langle X, Y \rangle_\sigma := \langle X, Y \rangle = \sum_{i=1}^{p} X_i Y_i.$$

(ii) $p \neq q$ and $t = lcm(p, q)$.

$$\langle X, Y \rangle_\sigma := \left\langle X \otimes \mathbf{1}_{t/p}, Y \otimes \mathbf{1} t/q \right\rangle .$$

$\mathbb{R}^\sigma$ becomes a dimension-free inner product space.

III. Equivalence of Matrices

Denote

$$\mathcal{M} = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{M}_{m\times n}.$$

Then $\ltimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, which is a monoid (i.e., a semi-group with identity).
Consider $A \in \mathcal{M}_{2\times 3}$, $B_i \in \mathcal{M}_{2i\times 3}$, $i = 1, 2, 3, \cdots$. By definition:

$$A \ltimes B_i = \begin{cases} (A \otimes I_2)(B_1 \otimes I_3), & i = 1 \\ (A \otimes I_4)(B_2 \otimes I_3), & i = 2 \\ (A \otimes I_8)(B_4 \otimes I_3), & i = 4 \\ \vdots & \end{cases}$$

☞ Equivalence Relation

It is easy to see that the STP is a product of two equivalences:

$$\langle A\rangle = \{A, A\otimes I_2, A\otimes I_3, \cdots\}, \quad \langle B\rangle = \{B, B\otimes I_2, B\otimes I_3, \cdots\}.$$

Definition 3.1

Let $A,\ B\in \mathcal{M}$ be two matrices. A and B are said to be equivalent, denoted by $A\sim B$, if there exist $I_s,\ I_t,\ s,t\in \mathbb{N}$, such that

$$A\otimes I_s = B\otimes I_t. \tag{34}$$

Denote

$$\langle A\rangle = \{B \mid B\sim A\}.$$

Theorem 3.2

(i) If $A \sim B$, then there exists a Λ such that

$$A = \Lambda \otimes I_\beta, \quad B = \Lambda \otimes I_\alpha. \tag{35}$$

(ii) In the equivalence class $\langle A \rangle_\ell$ there exists a unique $A_1 \in \langle A \rangle_\ell$, such that A_1 is irreducible. That is, there is no I_s, $s > 1$ such that

$$A = B \otimes I_s.$$

In (34), w.l.g., we assume $gcd(s,t) = 1$, then we define

$$\Theta := A \otimes I_s = B \otimes I_t. \tag{36}$$

☞ Lattice

Definition 3.3

Consider a set Q with a relation $\prec$.

1. $(Q, \prec)$ is called a partial order set, if
   - **(i)** (self-reflect) $a \prec a$;
   - **(ii)** (non-symmetric) if $a \prec b$ and $b \prec a$, then $a = b$;
   - **(iii)** (transitive) if $a \prec b$ and $b \prec c$, then $a \prec c$.
2. A partial order set $(Q, \prec)$ is called a total order set, if for any $a, b \in Q$ we have either $a \prec b$ or $b \prec a$, then $(Q, \prec)$ is called a total order set.

Definition 3.4

(i) Let Q be a partial order set and $A \subset Q$. $p \in Q$ is called an upper boundary of A, if $a \prec p$, $\forall a \in A$.

(ii) p is an upper boundary of A. p is called the least upper boundary of A, denoted by $p = \sup(A)$, if for any upper boundary u of A, $p \prec u$.

(iii) $q \in S$ is called a lower boundary of A, if $q \prec a$, $\forall a \in A$.

(iv) q is a lower boundary of A. q is called the greatest lower boundary of A, denoted by $q = \inf(A)$, if for any lower boundary ℓ of A, $\ell \prec q$.

Definition 3.5

A partial order set $(Q, \prec)$ is a lattice, if for any two elements $a,\ b \in Q$, there are $\sup\{a,\ b\}$ and $\inf\{a,\ b\}$.

☞ Lattice Structure of $\langle A \rangle$

Definition 3.6

Let $A,\ B \in \langle A \rangle$ $A \prec B$ if there exists I_k such that

$$A \otimes I_k = B.$$

Theorem 3.7

$(\langle A \rangle\ , \prec)$ is a lattice. For any $A, B \in \langle A \rangle$,

$$\sup(A, B) = \Theta; \quad \inf(A, B) = \Lambda,$$

where Θ and Λ are defined in (35) with $gcd(\alpha, \beta) = 1$ and in (36) with $gcd(s, t) = 1$ respectively.

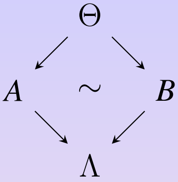

Figure 1: A Lattice Structure: $\Theta = \sup(A, B) \;\; \Lambda = \inf(A, B)$

Example 3.8

Let

$$\langle A \rangle = \{A_1, A_2 = A_1 \otimes I_2, A_3 = A_1 \otimes I_3, \cdots\}.$$

Then

$$\sup(A_4, A_6) = A_{12}; \quad \inf(A_4, A_6) = A_2.$$

☞ Quotient Space

Definition 3.9

Let $A, B \in \mathcal{M}$. Define

$$\langle A \rangle \ltimes \langle B \rangle := \langle A \ltimes B \rangle \,. \tag{37}$$

Proposition 3.10

(i) The class product (37) is well defined. That is, if $A \sim A'$ and $B \sim B'$, then

$$A \ltimes B \sim A' \ltimes B'. \tag{38}$$

(ii) The class product (37) is associative. That is,

$$(\langle A \rangle \ltimes \langle B \rangle) \ltimes \langle C \rangle = \langle A \rangle \ltimes (\langle B \rangle \ltimes \langle C \rangle). \tag{39}$$

Define the quotient space as

$$\Omega := \mathcal{M}/\sim .$$

Proposition 3.11

(i) $(\Omega, \ltimes)$ is a monoid (simi-group with identity).

(ii) Let $\mathcal{M}_1 = \{M \in \mathcal{M} \mid M \text{ is invertible}\}$, $\Omega_1 = \mathcal{M}_1/\sim$. Then $(\Omega_1, \ltimes)$ is a group.

Note that in Ω the identity element is

$$e = \langle 1 \rangle = \{I_n \mid n = 1, 2, 3, \cdots\}.$$

IV. Generalized STP

☞ Matrix Multiplier

Recalling the MM-L STP, for $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$

$$A \ltimes B = \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right). \tag{40}$$

Where the set

$$I = \{1, I_2, I_3, \cdots\}$$

are called a matrix multiplier.

Q: Can we find another set of matrices to replace this set?

Fundamental Requirements:

(i) Using this set, the new STP is a generalization of conventional matrix product.

(ii) The new STP is associative.

Definition 4.1

A set of matrices

$$\Gamma := \{\Gamma_n \in \mathcal{M}_{n\times n} \mid n \geq 1\}$$

is called a matrix multiplier, if

$$\Gamma_1 = 1; \tag{41}$$

$$\Gamma_n \Gamma_n = \Gamma_n; \tag{42}$$

$$\Gamma_p \otimes \Gamma_q = \Gamma_{pq}. \tag{43}$$

☞ Multiplier-based STP

Definition 4.2

Assume $\Gamma = \{\Gamma_n, \mid n \geq 1\}$ is a multiplier, $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$. Then the multiplier Γ based left STP of A and B is defined as

$$A \ltimes_{\Gamma} B := \left(A \otimes \Gamma_{t/n}\right)\left(B \otimes \Gamma_{t/p}\right). \tag{44}$$

where $t = lcm(n,p)$.
The multiplier Γ based right STP of A and B is defined as

$$A \rtimes_{\Gamma} B := \left(\Gamma_{t/n} \otimes A\right)\left(\Gamma_{t/p} \otimes B\right). \tag{45}$$

Example 4.3

- Assume $\Gamma = I := \{I_n\}$. It is a matrix multiplier. In fact,

$$\ltimes_\Gamma = \ltimes; \quad \rtimes_\Gamma = \rtimes.$$

- Set

$$J_n := \frac{1}{n}\mathbf{1}_{n\times n}, \quad n = 1, 2, \cdots . \tag{46}$$

It is easy to verify that $\Gamma = J := \{J_n \mid n = 1, 2, \cdots\}$ satisfies (41)-(43), hence, it is a matrix multiplier.

Example 4.3(cont'd)

- Set $\Delta_n^U \in \mathcal{M}_{n\times n}$ as:

$$\left(\Delta_n^U\right)_{i,j} = \begin{cases} 1, & i = 1, \text{ and } j = 1, \\ 0, & \text{Otherwise.} \end{cases} \tag{47}$$

  It is easy to verify that $\Delta^U := \{\Delta_n^U \mid n = 1, 2, \cdots\}$ satisfies (41)-(43), hence, it is a matrix multiplier.

- Set $\Delta_n^D \in \mathcal{M}_{n\times n}$ as:

$$\left(\Delta_n^D\right)_{i,j} = \begin{cases} 1, & i = n, \text{ and } j = n, \\ 0, & \text{Otherwise.} \end{cases} \tag{48}$$

  It is easy to verify that $\Delta^D := \{\Delta_n^D \mid n = 1, 2, \cdots\}$ satisfies (41)-(43), hence, it is a matrix multiplier.

☞ Second Matrix-Matrix (Second MM-L) STP

Using J defined by (46)，we define second MM-L STP:

Definition 4.4

Using $\Gamma = J = \{J_n \mid n = 1, 2, \cdots\}$, the second MM-L STP is defined as

(i) second MM-L STP:

$$A \circ_\ell B := \left(A \otimes J_{t/n}\right)\left(B \otimes J_{t/p}\right). \tag{49}$$

(ii) second MM-R STP:

$$A \circ_\ell B := \left(J_{t/n} \otimes A\right)\left(J_{t/p} \otimes B\right). \tag{50}$$

☞ Vector Multiplier

Definition 4.6

A vector sequence

$$\gamma : \{\gamma_r \in \mathbb{R}^n \mid r \geq 1\}$$

is called a vector multiplier, if it satisfies the following:

$$\gamma_1 = 1; \tag{51}$$

$$\gamma_p \otimes \gamma_q = \gamma_{pq}. \tag{52}$$

Example 4.7

(i)

$$\gamma = \mathbf{1} := \{\mathbf{1}_n \mid n = 1, 2, \cdots\}. \tag{53}$$

(ii)

$$\gamma = \delta^U := \{\delta_n^1 \mid n = 1, 2, \cdots\}. \tag{54}$$

(iii)

$$\gamma = \delta^D := \{\delta_n^n \mid n = 1, 2, \cdots\}. \tag{55}$$

Proposition 4.8

If $\gamma = \{\gamma_n \mid n = 1, 2, \cdots\}$ is a vector multiplier, then, $\gamma- = \{\gamma_{\cdot n} \mid n = 1, 2, \cdots\}$ is also a vector multiplier, where

$$\gamma_n' = n^k \gamma_n, \quad n = 1, 2, \cdots.$$

☞ Matrix-Vector (MV) STP

Definition 4.9

Let Γ be a matrix multiplier, γ a vector multiplier, $A \in \mathcal{M}_{m\times n}$, $x \in \mathbb{R}^r$, $t = n \vee r$. Then the matrix-vector STP of A and x related with Γ and γ, denoted by $\vec{\times}$, is defined as

- Left MV-STP:

$$A \vec{\times}_{\ell} x := \left(A \otimes \Gamma_{t/p}\right)\left(x \otimes \gamma_{t/r}\right). \tag{56}$$

- Right MV-STP:

$$A \vec{\times}_{r} x := \left(\Gamma_{t/p} \otimes A\right)\left(\gamma_{t/r} \otimes x\right). \tag{57}$$

☞ MM vs MV

Remark 4.10

(i) MM-STP is used for composition of two linear mappings.

(ii) MV-STP is used for realizing linear mapping.

(iii) In classical case, they are coincide.

☞ Two Important MV- STPs

Definition 4.11

MV-1 STP:

$$\Gamma = \{I_n \mid n = 1, 2, \cdots\}, \quad \gamma = \{\mathbf{1}_n \mid n = 1, 2, \cdots\}.$$

Let $A \in \mathcal{M}_{m\times n}$, $x \in \mathbb{R}^r$, $t = n \vee r$. Then,

(i) Left MV-1 STP:

$$A \vec{\ltimes} x := \left(A \otimes I_{t/p}\right)\left(x \otimes \mathbf{1}_{t/r}\right). \tag{58}$$

(ii) Right MV-1 STP:

$$A \vec{\rtimes} x := \left(I_{t/p} \otimes A\right)\left(\mathbf{1}_{t/r} \otimes x\right). \tag{59}$$

Definition 4.11(cont'd)

MV-2 STP:

$$\Gamma = \{J_n \mid n = 1, 2, \cdots\}, \quad \gamma = \{\mathbf{1}_n \mid n = 1, 2, \cdots\}.$$

Let $A \in \mathcal{M}_{m\times n}$, $x \in \mathbb{R}^r$, $t = n \vee r$. Then,

(i) Left MV-2 STP:

$$A \vec{\circ}_\ell x := \left(A \otimes J_{t/p}\right) \left(x \otimes \mathbf{1}_{t/r}\right). \tag{60}$$

(ii) Right MV-2 STP:

$$A \vec{\circ}_r x := \left(J_{t/p} \otimes A\right) \left(\mathbf{1}_{t/r} \otimes x\right). \tag{61}$$

V. Conclusion

☞ General Remark

- STP is a generalization of conventional matrix product, which keeps main properties of conventional matrix product available.
- STP has some further nice properties, such as pseudo-commutativity.
- STP makes $(\mathcal{M}, \ltimes)$ a semi-group.
- Choosing proper Matrix multiplier (Matrix and Vector multiplies), some new MM- (MV-) STP can be obtained.

$$\Gamma \;\Rightarrow\; \text{MM-STP}; \quad \Gamma + \gamma \;\Rightarrow\; \text{MV-STP}.$$

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VI. Appendix

☞ Proof of Associativity of STP

Consider

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C). \tag{62}$$

Assume $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, $C \in \mathcal{M}_{r\times s}$. Denote

$$\begin{array}{ll} lcm(n,p) = nn_1 = pp_1, & lcm(q,r) = qq_1 = rr_1, \\ lcm(r,qp_1) = rr_2 = qp_1p_2, & lcm(n,pq_1) = nn_2 = pq_1q_2. \end{array}$$

Then

$$
\begin{aligned}
(A \ltimes B) \ltimes C &= ((A \otimes I_{n_1})(B \otimes I_{p_1})) \ltimes C \\
&= (((A \otimes I_{n_1})(B \otimes I_{p_1})) \otimes I_{p_2})(C \otimes I_{r_2}) \\
&= (A \otimes I_{n_1 p_2})(B \otimes I_{p_1 p_2})(C \otimes I_{r_2}).
\end{aligned}
$$

$$
\begin{aligned}
A \ltimes (B \ltimes C) &= A \ltimes ((B \otimes I_{q_1})(C \otimes I_{r_1})) \\
&= (A \otimes I_{n_2}) (((B \otimes I_{q_1})(C \otimes I_{r_1})) \otimes I_{q_2}) \\
&= (A \otimes I_{n_2}) (B \otimes I_{q_1 q_2})(C \otimes I_{r_1 q_2})) .
\end{aligned}
$$

Hence, to prove (62), it is enough to prove the following three equations:

$$n_1 p_2 = n_2 \tag{63a}$$
$$p_1 p_2 = q_1 q_2 \tag{63b}$$
$$r_2 = r_1 q_2 \tag{63c}$$

Recall the associativity of least common multiplier [?]:

$$lcm(i, lcm(j,k)) = lcm(lcm(i,j),k), \quad i,j,k \in \mathbb{N}, \tag{64}$$

[6] L. Hua, *An Introduction to Number Theory*, Science Press, Beijing, 1979 (in Chinese).

It follows that

$$lcm(qn, lcm(pq, pr)) = lcm(lcm(qn, pq), pr). \qquad (65)$$

Using (65), we have

$$\begin{aligned} \text{LHS of (63b)} &= lcm(qn, plcm(q, r)) \\ &= lcm(qn, pqq_1) \\ &= qlcm(n, pq_1) \\ &= qpq_1q_2. \end{aligned}$$

$$\begin{aligned} \text{RHS of (63b)} &= lcm(qlcm(n, p), pr) \\ &= lcm(qpp_1, pr) \\ &= plcm(qp_1, r) \\ &= pqp_1p_2. \end{aligned}$$

(63b) follows.

Using (63b), we have

$$\begin{aligned} n_1 p_2 &= n_1 \frac{q_1 q_2}{p_1} = n_1 \frac{q_1 q_2 p}{p_1 p} \\ &= \frac{lcm(n,p)}{n} \frac{lcm(n,pq_1)}{pp_1} \\ &= \frac{lcm(n,pq_1)}{n} = n_2, \end{aligned}$$

which shows (63a).
Similarly,

$$\begin{aligned} r_1 q_2 &= r_1 \frac{p_1 p_2}{q_1} = t_1 \frac{p_1 p_2 q}{q_1 q} \\ &= \frac{lcm(q,r)}{r} \frac{lcm(r,qp_1)}{q_1 q} \\ &= \frac{lcm(r,qp_1)}{r} = r_2, \end{aligned}$$

which shows (63c).

谢　谢　！

Any Question?