State Space, System Decomposition and Disturbance Decoupling of Boolean Networks

Series One, Lesson Six

Lecturer: Jiandong Zhu

(School of Mathematical Sciences, Nanjing Normal University, Nanjing)

Center of STP Theory and Its Applications
August 15-23, 2020

LiaoCheng University, LiaoCheng, Shandong, P.R. China

Outline

1. The Algebraic Model of Boolean Control Networks
2. Coordinate Transformation of BCNs
3. State Space and Regular subspaces
4. Decomposition w.r.t. Inputs
5. Disturbance Decoupling
6. Conclusion

I. The Algebraic Model of Boolean Control Networks

☞ Boolean Control Networks (BCNs)

A Boolean control network is described as

$$X(t+1) = f(X(t), U(t)), \quad (1)$$

$$Y(t) = h(X(t)), \quad (2)$$

where $X = (X_1, X_2, \ldots, X_n)^{\mathrm{T}} \in \mathcal{D}^n$ is the state vector, $U = (U_1, U_2, \ldots, U_p)^{\mathrm{T}} \in \mathcal{D}^p$ is the input vector, $Y = (Y_1, Y_2, \ldots, Y_q)^{\mathrm{T}} \in \mathcal{D}^q$ is the output vector, f and h are logical mapping, and $\mathcal{D} = \{0, 1\}$.

What is the difference between BCNs and traditional control systems?

I. The Algebraic Model of Boolean Control Networks

☞ Boolean Control Networks (BCNs)

A Boolean control network is described as

$$X(t+1) = f(X(t), U(t)), \quad (1)$$

$$Y(t) = h(X(t)), \quad (2)$$

where $X = (X_1, X_2, \ldots, X_n)^{\mathrm{T}} \in \mathcal{D}^n$ is the state vector, $U = (U_1, U_2, \ldots, U_p)^{\mathrm{T}} \in \mathcal{D}^p$ is the input vector, $Y = (Y_1, Y_2, \ldots, Y_q)^{\mathrm{T}} \in \mathcal{D}^q$ is the output vector, f and h are logical mapping, and $\mathcal{D} = \{0, 1\}$.

What is the difference between BCNs and traditional control systems?

Example 1

$$\begin{aligned} X_1(t+1) &= U(t) \to (X_2(t) \wedge X_3(t)), \\ X_2(t+1) &= \neg X_1(t), \\ X_3(t+1) &= X_2(t) \vee X_3(t), \\ Y(t) &= (X_1(t) \leftrightarrow X_2(t)) \vee \neg X_3(t). \end{aligned} \quad (3)$$

Example 2

The λ–Switch genetic regulation network is described by

$$\begin{cases} N(k+1) = (\neg cI(k)) \wedge (\neg cro(k)), \\ cI(k+1) = (\neg cro(k)) \wedge (cI(k) \vee cII(k)), \\ cII(k+1) = (\neg cI(k)) \wedge u(k) \wedge (N(k) \vee cIII(k)), \\ cIII(k+1) = (\neg cI(k)) \wedge u(k) \wedge N(k), \\ cro(k+1) = (\neg cI(k)) \wedge (\neg cII(k)). \end{cases} \quad (4)$$

Example 1

$$\begin{aligned} X_1(t+1) &= U(t) \rightarrow (X_2(t) \wedge X_3(t)), \\ X_2(t+1) &= \neg X_1(t), \\ X_3(t+1) &= X_2(t) \vee X_3(t), \\ Y(t) &= (X_1(t) \leftrightarrow X_2(t)) \vee \neg X_3(t). \end{aligned} \tag{3}$$

Example 2

The $\lambda-$Switch genetic regulation network is described by

$$\begin{cases} N(k+1) = (\neg cI(k)) \wedge (\neg cro(k)), \\ cI(k+1) = (\neg cro(k)) \wedge (cI(k) \vee cII(k)), \\ cII(k+1) = (\neg cI(k)) \wedge u(k) \wedge (N(k) \vee cIII(k)), \\ cIII(k+1) = (\neg cI(k)) \wedge u(k) \wedge N(k), \\ cro(k+1) = (\neg cI(k)) \wedge (\neg cII(k)). \end{cases} \tag{4}$$

☞ The Algebraic Form of BCNs

Lemma 1 (Cheng & Qi, 2010a). Let $x = \ltimes_{i=1}^{n} x_i$ and $u = \ltimes_{i=1}^{p} u_i$. Then the Boolean control system can be expressed in the algebraic form

$$x(t+1) = Lu(t)x(t) = [L_1,\ L_2,\ \ldots,\ L_{2^p}]u(t)x(t) \qquad (5)$$

$$y(t) = Hx(t). \qquad (6)$$

where L_i, L and H are $2^n \times 2^n$, $2^n \times 2^{n+p}$ and $2^q \times 2^n$ logical matrices, respectively.

[1] Cheng, D., & Qi, H. (2009). Controllability and observability of Boolean control networks. *Automatica*, 45(7), 1659–1667.

Graphic meaning of the algebraic form of a BCN

Consider a Boolean control network with $n = 3$, $p = 1$ and $q = 1$ as follows:

$$\begin{aligned} x(t+1) &= Lu(t)x(t) = [L_1, \;\; L_2]u(t)x(t), \\ y(t) &= Hx_1(t). \end{aligned} \tag{7}$$

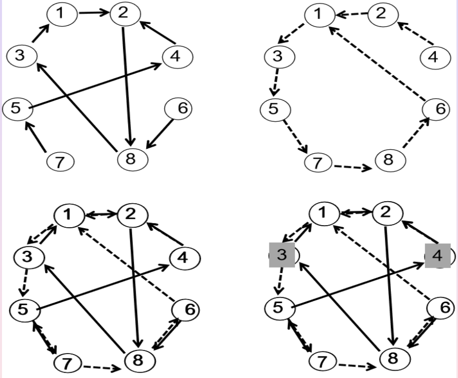


Figure 1: The state transition graph of the BCN.

☞ Coordinate Transformation of BCNs)

Definition 1

A transformation $Z = g(X)$ is called a *logical coordinate transformation* if $g: \mathcal{D}^n \to \mathcal{D}^n$ is a bijection.

Lemma

Let $z = Gx$ be the algebraic form of the logical coordinate transformation $Z = g(X)$, where G is a permutation matrix (invertible logical matrix). Then BCN (9) becomes

$$\begin{cases} z(t+1) = GL(I_p \otimes G^{\mathrm{T}})u(t)z(t), \\ y(t) = HG^{\mathrm{T}}z(t). \end{cases} \tag{8}$$

Proof.

$$\begin{aligned} z(t+1) &= Gx(t+1) = GLu(t)x(t), \\ &= GLu(t)G^{\mathrm{T}}z(t) = GL(I_p \otimes G^{\mathrm{T}})u(t)z(t). \end{aligned} \tag{9}$$

☞ Coordinate Transformation of BCNs)

Definition 1

A transformation $Z = g(X)$ is called a *logical coordinate transformation* if $g: \mathcal{D}^n \to \mathcal{D}^n$ is a bijection.

Lemma

Let $z = Gx$ be the algebraic form of the logical coordinate transformation $Z = g(X)$, where G is a permutation matrix (invertible logical matrix). Then BCN (9) becomes

$$\begin{cases} z(t+1) = GL(I_p \otimes G^{\mathrm{T}})u(t)z(t), \\ y(t) = HG^{\mathrm{T}}z(t). \end{cases} \tag{8}$$

Proof.

$$\begin{aligned} z(t+1) &= Gx(t+1) = GLu(t)x(t), \\ &= GLu(t)G^{\mathrm{T}}z(t) = GL(I_p \otimes G^{\mathrm{T}})u(t)z(t). \end{aligned} \tag{9}$$

Lemma

Let S be a $m \times n$ matrix with non-negative integral entries. Then S is a logical matrix if and only if

$$\mathbf{1}_m^{\mathrm{T}} S = \mathbf{1}_n.$$

Lemma

Let R be an $n \times n$ logical matrix. Then R is a permutation matrix if and only if

$$R\mathbf{1}_n = \mathbf{1}_n.$$

Lemma

Let A, B and C be matrices with appropriate sizes. Then

$$V_c(ABC) = (C^{\mathrm{T}} \otimes A)V_c(B),$$

where V_c is the column-stacking operator.

Lemma

Set A, B, C, D have proper dimensions, then (Horn & Johnson, 1989)

$$AC \otimes BD = (A \otimes B)(C \otimes D).$$

Set $X \in \triangle_m, Y \in \triangle_n$. Then $\mathbf{1}_m^\mathrm{T} X = 1$ and $\mathbf{1}_n^\mathrm{T} Y = 1$. So we can easily get the results as follows:

Lemma

If $X \in \triangle_m, Y \in \triangle_n$, then

$$X = (I_m X) \otimes (\mathbf{1}_n^\mathrm{T} Y) = (I_m \otimes \mathbf{1}_n^\mathrm{T})(XY), \quad (10)$$

$$Y = (\mathbf{1}_m^\mathrm{T} X) \otimes (I_n Y) = (\mathbf{1}_m^\mathrm{T} \otimes I_n)(XY). \quad (11)$$

II. State Space and Regular subspaces

Definition

The state space $\mathcal{X}$ is defined as the set of all logical functions with respect to $\{x_1, x_2, \cdots, x_n\}$ denoted by $\mathcal{F}_l\{x_1, x_2, \cdots, x_n\}$. Let $z_1, z_2, \cdots, z_k \in \mathcal{X}$. The subspace generated by $\{z_1, z_2, \cdots, z_k\}$ is defined as the set of logical functions with respect to $\{z_1, z_2, \cdots, z_k\}$, denoted by $\mathcal{Z} = \mathcal{F}_l\{z_1, z_2, \cdots, z_k\}$.

Example 4

Assume the state space is $\mathcal{X} = \mathcal{F}_l\{x_1, x_2, x_3\}$. Let $z_1 = x_1 \vee x_2$ and $z_2 = \neg x_3 \wedge x_1$. $\mathcal{Z} = \mathcal{F}_l\{z_1, z_2\}$ is a subspace. Since each $z \in \mathcal{Z}$ can be regarded as a composite function with respect to $\{x_1, x_2, x_3\}$, we write that $\mathcal{Z} \subset \mathcal{X}$.

II. State Space and Regular subspaces

Definition

The state space $\mathcal{X}$ is defined as the set of all logical functions with respect to $\{x_1, x_2, \cdots, x_n\}$ denoted by $\mathcal{F}_l\{x_1, x_2, \cdots, x_n\}$. Let $z_1, z_2, \cdots, z_k \in \mathcal{X}$. The subspace generated by $\{z_1, z_2, \cdots, z_k\}$ is defined as the set of logical functions with respect to $\{z_1, z_2, \cdots, z_k\}$, denoted by $\mathcal{Z} = \mathcal{F}_l\{z_1, z_2, \cdots, z_k\}$.

Example 4

Assume the state space is $\mathcal{X} = \mathcal{F}_l\{x_1, x_2, x_3\}$. Let $z_1 = x_1 \vee x_2$ and $z_2 = \neg x_3 \wedge x_1$. $\mathcal{Z} = \mathcal{F}_l\{z_1, z_2\}$ is a subspace. Since each $z \in \mathcal{Z}$ can be regarded as a composite function with respect to $\{x_1, x_2, x_3\}$, we write that $\mathcal{Z} \subset \mathcal{X}$.

Definition

A subspace $\mathcal{Z} = \mathcal{F}_l\{z_1, z_2, \cdots, z_k\}$ is called a regular subspace of dimension k if there are $z_{k+1}, z_{k+2}, \cdots, z_n \in \mathcal{X}$ such that $Z = (z_1, z_2, \cdots, z_n)^{\mathrm{T}}$ is a logical coordinate transformation. $\{z_1, z_2, \cdots, z_k\}$ is called a sub-basis of $\mathcal{Z}$. $\mathcal{F}_l\{z_{k+1}, z_{k+2}, \cdots, z_n\}$ is called the complementary space of $\mathcal{Z}$.

BCNs V.S. Traditional Linear Control Systems

BCNs

- BCN: $x(t+1) = Lu(t)x(t)$
- logical coordinate transformation: $z = Gx$
- Equivalent BCN:

$$z(t+1) = GL(I_p \otimes G^{\mathrm{T}})u(t)z(t)$$

Traditional Linear Control Systems

1. System model: $x(t+1) = Ax(t) + Bu(t)$
2. Coordinate transformation: $z = Tx$
3. Equivalent system:

$$z(t+1) = TAT^{-1}z(t) + TBu(t)$$

BCNs V.S. Traditional Linear Control Systems

BCNs

- BCN: $x(t+1) = Lu(t)x(t)$
- logical coordinate transformation: $z = Gx$
- Equivalent BCN:

$$z(t+1) = GL(I_p \otimes G^{\mathrm{T}})u(t)z(t)$$

Traditional Linear Control Systems

1. System model: $x(t+1) = Ax(t) + Bu(t)$
2. Coordinate transformation: $z = Tx$
3. Equivalent system:

$$z(t+1) = TAT^{-1}z(t) + TBu(t)$$

BCNs V.S. Linear Control Systems

BCNs

- **Problem**: Given s logical functions $z_1, z_2, \cdots, z_s$ with the algebraic form

  $$\bar{z} = z_1 z_2 \cdots z_s = Mx,$$

  are there $n-s$ logical functions $z_{s+1}, z_{s+2}, \ldots, z_n$ with the algebraic form $\tilde{z} = z_{s+1} z_{s+2} \cdots z_n = Nx$ such that $z = \bar{z}\tilde{z} = MxNx$ is a logical coordinate transformation?

Linear Control Systems

1. **Problem**: Given

   $$\bar{z} = [z_1, z_2, \ldots, z_s]^{\mathrm{T}} = Mx,$$

   are there $n-s$ functions $z_{s+1}, z_{s+2}, \ldots, z_n$ with the form $\tilde{z} = [z_{s+1}, z_{s+2}, \ldots, z_n]^{\mathrm{T}} = Nx$ such that

   $$z = \begin{bmatrix} \bar{z} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} x$$

   is a coordinate transformation?

BCNs V.S. Linear Control Systems

BCNs

- **Problem**: Given s logical functions $z_1, z_2, \cdots, z_s$ with the algebraic form

$$\bar{z} = z_1 z_2 \cdots z_s = Mx,$$

are there $n-s$ logical functions $z_{s+1}, z_{s+2}, \ldots, z_n$ with the algebraic form $\tilde{z} = z_{s+1} z_{s+2} \cdots z_n = Nx$ such that $z = \bar{z}\tilde{z} = MxNx$ is a logical coordinate transformation?

Linear Control Systems

1. **Problem**: Given

$$\bar{z} = [z_1, z_2, \ldots, z_s]^{\mathrm{T}} = Mx,$$

are there $n-s$ functions $z_{s+1}, z_{s+2}, \ldots, z_n$ with the form $\tilde{z} = [z_{s+1}, z_{s+2}, \ldots, z_n]^{\mathrm{T}} = Nx$ such that

$$z = \begin{bmatrix} \bar{z} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} x$$

is a coordinate transformation?

Regular Subspaces

Theorem

Let $\mathcal{X} = \mathcal{F}_l(x_1, x_2, \cdots, x_n)$ *be the state space. Consider a subspace* $\overline{\mathcal{Z}} = \mathcal{F}_l(z_1, z_2, \cdots, z_s) \subset \mathcal{X}$. *Let*

$$x = x_1 x_2 \cdots x_n, \quad \bar{z} = z_1 z_2 \cdots z_s = Mx,$$

where M *is a logical matrix. Then* $\overline{\mathcal{Z}}$ *is a regular subspace with sub-basis* $z_1, z_2, \cdots, z_s$ *if and only if there exists a logical matrix* N *such that*

$$MN^{\mathrm{T}} = \mathbf{1}_{2^s, 2^{n-s}}, \tag{12}$$

where $\mathbf{1}_{2^s, 2^{n-s}}$ *denotes the* $2^s \times 2^{n-s}$ *matrix of ones.*

Proof. Let $z_{s+1}, z_{s+2}, \cdots, z_n \in \mathcal{X}$ be $n-s$ logical functions with the algebraic form $\tilde{z} = z_{s+1}z_{s+2}\cdots z_n = Nx$. Then $\overline{\mathcal{Z}}$ is a regular subspace with sub-basis $z_1, z_2, \cdots, z_s$ if and only if there exists a logical matrix N such that the transformation $z = \bar{z}\tilde{z} = MxNx$ is a logical coordinate transformation. A straightforward calculation shows that

$$\begin{aligned} MxNx &= M(I_{2^n} \otimes N)xx = M(I_{2^n} \otimes N)\Phi_{2^n}x \\ &= (M \otimes I_{2^{n-s}})(I_{2^n} \otimes N)\Phi_{2^n}x \\ &= (M \otimes N)\Phi_{2^n}x. \end{aligned} \tag{13}$$

Thus, $z = MxNx$ is a logical coordinate transformation if and only if the transformation matrix $(M \otimes N)\Phi_{2^n}$ is a permutation matrix, that is,

$$(M \otimes N)\Phi_{2^n}\mathbf{1}_{2^n} = \mathbf{1}_{2^n}. \tag{14}$$

A straightforward computation shows that

$$\Phi_{2^n}\mathbf{1}_{2^n}=\begin{bmatrix}\delta_{2^n}^1 & & & \\ & \delta_{2^n}^2 & & \\ & & \ddots & \\ & & & \delta_{2^n}^{2^n}\end{bmatrix}\begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}=\begin{bmatrix}\delta_{2^n}^1\\ \delta_{2^n}^2\\ \vdots\\ \delta_{2^n}^{2^n}\end{bmatrix}=V_c(I_{2^n}). \tag{15}$$

Substituting (15) into (14) yields

$$(M\otimes N)V_c(I_{2^n})=\mathbf{1}_{2^n}. \tag{16}$$

By Lemma 4, (16) can be rewritten as $NM^{\mathrm{T}}=\mathbf{1}_{2^{n-s},2^s}$, which is just (12). Thus the theorem is proved.

Theorem

With the conditions and notions of the above theorem, $\overline{\mathcal{Z}}$ is a regular subspace with sub-basis $z_1, z_2, \cdots, z_s$ if and only if

$$M\mathbf{1}_{2^n} = 2^{n-s}\mathbf{1}_{2^s} \tag{17}$$

or equivalently

$$MM^{\mathrm{T}} = 2^{n-s}I_{2^s} \tag{18}$$

Proof. (Necessity) Multiplying (12) on the right by $\mathbf{1}_{2^{n-s}}$ yields

$$MN^{\mathrm{T}}\mathbf{1}_{2^{n-s}} = 2^{n-s}\mathbf{1}_{2^s}. \tag{19}$$

Considering N is a logical matrix, we have that $N^{\mathrm{T}}\mathbf{1}_{2^{n-s}} = \mathbf{1}_{2^n}$. Thus (17) follows from (19).

(Sufficiency) Since (17) holds, each row of M has exact 2^{n-s} nonzero elements equal to 1. Thus there exists a permutation matrix T such that

$$MT = \mathbf{1}_{2^{n-s}}^{\mathrm{T}} \otimes I_{2^s}. \quad (20)$$

Choose

$$N = (I_{2^{n-s}} \otimes \mathbf{1}_{2^s}^{\mathrm{T}})T^{\mathrm{T}}. \quad (21)$$

A straightforward computation shows that N is a logical matrix and

$$MN^{\mathrm{T}} = (\mathbf{1}_{2^{n-s}}^{\mathrm{T}} \otimes I_{2^s})(I_{2^{n-s}} \otimes \mathbf{1}_{2^s}) = \mathbf{1}_{2^{n-s}}^{\mathrm{T}} \otimes \mathbf{1}_{2^s} = \mathbf{1}_{2^s, 2^{n-s}}. \quad (22)$$

Thus, by Theorem 9, $\overline{\mathcal{Z}}$ is a regular subspace with subbasis $z_1, z_2, \cdots, z_s$.

Example 1. Consider the third-order BCN

$$\begin{cases} x_1(t+1) = \neg(x_1(t) \leftrightarrow x_2(t)), \\ x_2(t+1) = \neg(x_2(t) \leftrightarrow x_3(t)), \\ x_3(t+1) = x_1(t) \wedge u(t), \\ y(t) = x_1(t) \leftrightarrow x_2(t), \end{cases}$$

where x_1, x_2 and x_3 are the states, u the control and y the output. Consider the subspace $\overline{\mathcal{Z}} = \mathcal{F}_l(z_1, z_2)$ of the state space $\mathcal{X} = \mathcal{F}_l(x_1, x_2, x_3)$, where

$$\begin{cases} z_1(t) = x_1(t) \leftrightarrow x_2(t), \\ z_2(t) = x_2(t) \bar{\vee} x_3(t)), \end{cases}$$

or the algebraic form

$$\bar{z}(t) = z_1(t) z_2(t) = Mx,$$

with $M = \delta_4[2\ 1\ 3\ 4\ 4\ 3\ 1\ 2]$. It is easy to check that $M\mathbf{1}_8 = 2\mathbf{1}_4$. Thus $\overline{\mathcal{Z}}$ is a regular subspace.

Since $M = \delta_4[2\ 1\ 3\ 4\ 4\ 3\ 1\ 2]$, we can let

$$T = \delta_8[2\ 1\ 3\ 4\ 7\ 8\ 6\ 5]$$

satisfying

$$MT = \delta_4[1\ 2\ 3\ 4\ 1\ 2\ 3\ 4]. \tag{23}$$

By (21), we get

$$\begin{aligned} N &= (I_2 \otimes \mathbf{1}_4^{\mathrm{T}})T^{\mathrm{T}} \\ &= \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2]\delta_8[2\ 1\ 3\ 4\ 8\ 7\ 5\ 6], \\ &= \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2], \end{aligned} \tag{24}$$

which implies that $z_3 = Nx = x_1$. Thus a logical coordinate transformation is obtained as

$$\begin{aligned} z_1(t) &= x_1(t) \leftrightarrow x_2(t), \\ z_2(t) &= x_2(t) \bar{\vee} x_3(t), \\ z_3(t) &= x_1(t). \end{aligned}$$

It should be noted that the complementary subspace of $\overline{\mathcal{Z}}$ is not unique. From $M = \delta_4[2\ 1\ 3\ 4\ 4\ 3\ 1\ 2]$, we can get another permutation matrix solution

$$T = \delta_8[7\ 1\ 6\ 4\ 2\ 8\ 3\ 5]$$

satisfying

$$MT = \delta_4[1\ 2\ 3\ 4\ 1\ 2\ 3\ 4]. \quad (25)$$

Then

$$\begin{aligned} N &= (I_2 \otimes \mathbf{1}_4^{\mathrm{T}})T^{\mathrm{T}} \\ &= \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2]\delta_8[2\ 5\ 7\ 4\ 8\ 3\ 1\ 6], \\ &= \delta_2[1\ 2\ 2\ 1\ 2\ 1\ 1\ 2], \end{aligned} \quad (26)$$

which yields $z_3 = (x_1 \wedge (x_2 \leftrightarrow x_3)) \vee (\neg x_1 \wedge (x_2 \bar{\vee} x_3))$.

☞ Regular Subspaces and Equal Vertex Partitions

Let $\overline{\mathcal{Z}}$ be a regular subspace with regular basis $\bar{z} = Mx$. Assume that the complementary subspace of $\overline{\mathcal{Z}}$ is $\overline{\mathcal{Z}}$ has the regular basis $\tilde{z} = Nx$. We have revealed the relationship between M and N is essentially

$$MN^{\mathrm{T}} = \mathbf{1}_{2^s,2^{n-s}}.$$

What is the graphical meaning?

Actually, M is corresponding to an equal vertex partition $\mathcal{P}_M = \{P_i^M\}_{i=1}^{2^s}$ with $|P_i^M| = 2^{n-s}$, N is corresponding to an equal vertex partition $\mathcal{P}_N = \{P_j^N\}_{j=1}^{2^{n-s}}$ with $|P_j^M| = 2^s$, and $|P_i^N \bigcap P_j^N| = 1$ for any i and j.

Example

$$M = \delta_4[2\ 1\ 3\ 4\ 4\ 3\ 1\ 2] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$N = \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

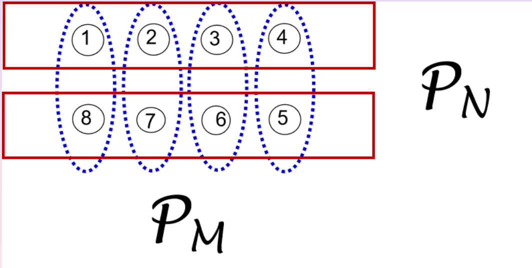


Figure 2: Coordinate transformation $z = \bar{z}\tilde{z} = MxNx$

Example

$$M = \delta_4[2\ 1\ 3\ 4\ 4\ 3\ 1\ 2] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$N = \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

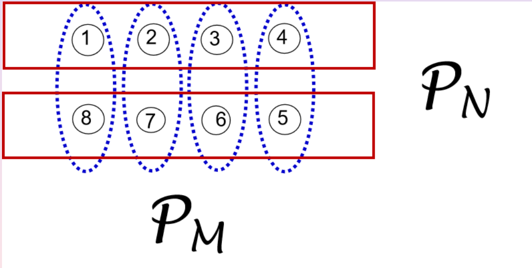


Figure 2: Coordinate transformation $z = \bar{z}\tilde{z} = MxNx$

Compare to the traditional linear control systems

For the second order linear control system

$$\dot{x} = Ax + Bu,$$

consider the linear coordinate transformation

$$\begin{bmatrix} \bar{z} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} Mx \\ Nx \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

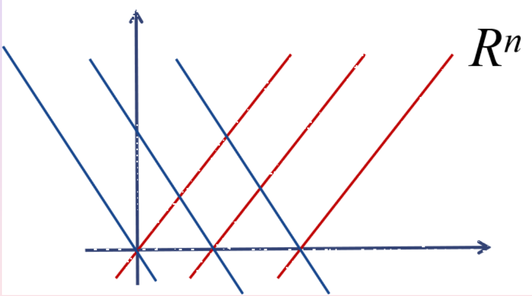


Figure 3: Linear coordinate transformation

☞ Controllability Decomposition

Consider linear control system

$$\dot{x} = Ax + Bu,$$

where $x \in R^n$ is the state, $u \in R^p$ is the control. Let $T = [\mathcal{X}_C \ \ U]$ be a nonsingular matrix, where $\mathcal{X}_C$ is a basis of the controllable subspace. Then under the transformation $z = T^{-1}x$, the system is equivalently transformed into the following system (Controllability Decomposition):

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \qquad (27)$$

where $(A_{11},\ B_1)$ is controllable, $z_1 \in R^{n_1}$.

controllability decomposition

The decomposition of separating uncontrollable subsystems

Consider a transformation $\tilde{z} = \tilde{T}^{-1}x$. Assume the system is equivalently into

$$\begin{bmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} u, \qquad (28)$$

where $z_1 \in R^{\tilde{n}_1}$.

An equivalent description of the controllability decomposition

Any controllability decomposition is one of the above decomposition with the minimum $\tilde{n}_1$.

Decomposition with respect to inputs

Consider BCN

$$x(t+1) = Lu(t)x(t). \tag{29}$$

If there exists a logical coordinate transformation

$$z_i = g_i(x_1, \cdots, x_n), \quad (i = 1, 2, \cdots, n)$$

with algebraic form $z = Tx$ such that the Boolean control system is equivalently transformed into

$$z^{[1]}(t+1) = G_1 u(t) z(t), \tag{30}$$

$$z^{[2]}(t+1) = G_2 z^{[2]}(t), \tag{31}$$

where G_1 and G_2 are $2^s \times 2^{n+m}$ and $2^{n-s} \times 2^{n-s}$ matrices with $s < n$ respectively, we say that the Boolean control system is decomposable with respect to inputs of order $n - s$.

Our Main Results

Definition 1

An equal vertex partition $\{S_l\}_{l=1}^{\mu}$ of the vertex set $A = \{1, 2, \cdots, 2^n\}$ is called a perfect equal vertex partition (PE-VP) if for any $l = 1, 2, \cdots, \mu$, there exists an α_i such that $\mathcal{N}(S_l) \subset S_{\alpha_l}$.

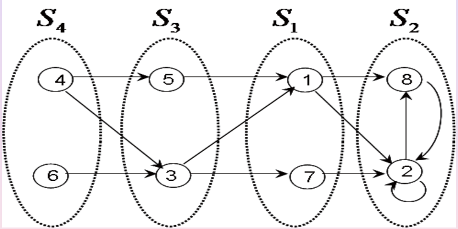


Figure 4: Perfect equal vertex partition.

Decomposability w.r.t. inputs

Theorem 2

The BCN is decomposable with respect to inputs with order $n-s$ if and only if the induced digraph $\mathcal{G}$ has a PE-VP $\{S_i\}_{i=1}^{2^{n-s}}$ with $|S_i| = 2^s$.

Problem

How can we get a perfect equal vertex partition (PE-VP) of the given directed graph? A straightforward method is check all the equal vertex partition one by one. But it is not realistic. Can we get some necessary conditions?

Decomposability w.r.t. inputs

Theorem 2

The BCN is decomposable with respect to inputs with order $n-s$ if and only if the induced digraph $\mathcal{G}$ has a PE-VP $\{S_i\}_{i=1}^{2^{n-s}}$ with $|S_i| = 2^s$.

Problem

How can we get a perfect equal vertex partition (PE-VP) of the given directed graph? A straightforward method is check all the equal vertex partition one by one. But it is not realistic. Can we get some necessary conditions?

An Algorithm for a PEVP

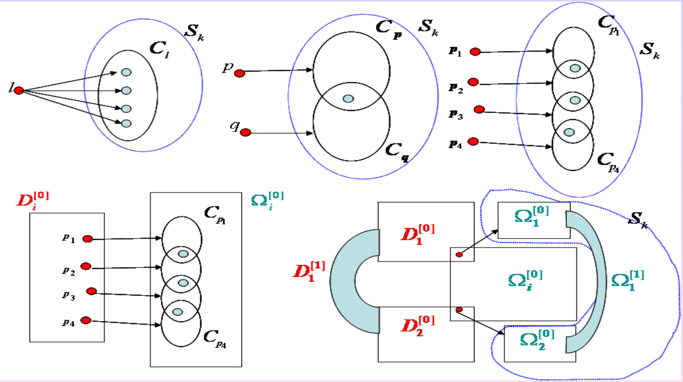


Figure 5: Necessary conditions for PEVP.

Vertex Set Uniting Algorithm (VSUA)

Algorithm

Using the obtained necessary conditions, we designed an algorithm called *Vertex Set Uniting Algorithm* (VSUA).

Vertex Set Uniting Algorithm (VSUA)

Step 0. Let $s_0 = s$, $D_i^{[0]} = D_i$ and $\Omega_i^{[0]} = \Omega_i$ for $i = 1, 2, \ldots, s_0$. Let $H_i^{[0]} = \{l \,|D_l^{[0]} \cap \Omega_i^{[0]} \neq \emptyset\}$.

- If $|H_i^{[0]}| = 1$ for every $i = 1, 2, \ldots, s_0$, then stop.
- Else there exists q_0 such that $|H_{q_0}^{[0]}| > 1$. Go to the next step.

Step μ. Let

$$D_1^{[\mu]} = \bigcup_{l \in H_{q_{\mu-1}}^{[\mu-1]}} D_l^{[\mu-1]}, \qquad \Omega_1^{[\mu]} = \bigcup_{l \in H_{q_{\mu-1}}^{[\mu-1]}} \Omega_l^{[\mu-1]}.$$

Relabel all the $D_l^{[\mu-1]}$ ($l \notin H_{q_{\mu-1}}^{[\mu-1]}$) by $D_i^{[\mu]}$ ($i = 2, 3, \ldots, s_\mu$). Similarly, rewrite all the $\Omega_l^{[\mu-1]}$ ($l \notin H_{q_{\mu-1}}^{[\mu-1]}$) by $\Omega_i^{[\mu]}$ ($i = 2, 3, \ldots, s_\mu$).

Vertex Set Uniting Algorithm (VSUA)

Let

$$H_i^{[\mu]} = \{l \,|D_l^{[\mu]} \cap \Omega_i^{[\mu]} \neq \emptyset\} \tag{32}$$

for every $i = 1, 2, \ldots, s_\mu$.
If $|H_i^{[\mu]}| = 1$ for every $i = 1, 2, \ldots, s_\mu$, then stop.
Else there exists q_μ such that $|H_{q_\mu}^{[\mu]}| > 1$. Go to the next step.

Theorem

The algorithm VSUA must stop in a finite number of steps. Suppose that $\mathcal{P} = \{P_i\}_{i=1}^{\tau}$ is a P-VP of $\mathcal{G}$ and the VSUA stops at step ρ. Then, for every $i = 1, 2, \ldots, s_\rho$, there exists an α_i and a β_i such that

$$\Omega_i^{[\rho]} \subset P_{\alpha_i}, \quad \Omega_i^{[\rho]} \subset D_{\beta_i}^{[\rho]}. \tag{33}$$

considers the BN described by

$$\begin{array}{rcl} X(t+1) & = & f(X(t), W(t)), \\ Y(t) & = & h(X(t)). \end{array} \tag{34}$$

For BNs, the first definition of disturbance decoupling is stated as follows:

Disturbance Decoupling

The disturbance decoupling is said to be implemented if there is a logical coordinate transformation $Z = \phi(X)$ such that under the Z coordinate frame the BN becomes

$$\begin{array}{rcl} Z^1(t+1) & = & \tilde{f}_1(Z(t), W(t)), \\ Z^2(t+1) & = & \tilde{f}_2(Z^2(t)), \\ Y(t) & = & \tilde{h}(Z^2(t)), \end{array} \tag{35}$$

where $Z^1(t) \in \mathcal{D}^{n_1}$, $Z^2(t) \in \mathcal{D}^{n_2}$, $Z = [Z^{1^{\mathrm{T}}}, Z^{2^{\mathrm{T}}}]^{\mathrm{T}} \in \mathcal{D}^n$, $n_1 + n_2 = n$, $\tilde{f}_i$ $i \in \{1, 2\}$ and $\tilde{h}$ are system mappings and output mapping, respectively.

Theorem

For BN (34), the disturbance decoupling described by the above definition is implemented if and only if the vertex-colored state transition graph of (34) has an ECP-VP.

The defined disturbance decoupling of BNs is dependent on whether systems can be decomposed into the form of (35).

Theorem

For BN (34), the disturbance decoupling described by the above definition is implemented if and only if the vertex-colored state transition graph of (34) has an ECP-VP.

The defined disturbance decoupling of BNs is dependent on whether systems can be decomposed into the form of (35).

However, recalling the original definition of disturbance decoupling of traditional linear control systems, we find that the concept of disturbance decoupling just means that the disturbance signals have no influence on outputs, which is actually independent of system decomposition. Motivated by this point, we introduced the concept of original disturbance decoupling as follows:

Original disturbance decoupling

Consider BN (34). The original disturbance decoupling is said to be implemented if, for each initial state $X(0) \in \mathcal{D}^n$, the output sequence $\{Y(s)\}_{s=0}^{+\infty}$ is the same for every disturbance sequence $\{W(s)\}_{s=0}^{+\infty}$ with each $W(s) \in \mathcal{D}^m$.

However, recalling the original definition of disturbance decoupling of traditional linear control systems, we find that the concept of disturbance decoupling just means that the disturbance signals have no influence on outputs, which is actually independent of system decomposition. Motivated by this point, we introduced the concept of original disturbance decoupling as follows:

Original disturbance decoupling

Consider BN (34). The original disturbance decoupling is said to be implemented if, for each initial state $X(0) \in \mathcal{D}^n$, the output sequence $\{Y(s)\}_{s=0}^{+\infty}$ is the same for every disturbance sequence $\{W(s)\}_{s=0}^{+\infty}$ with each $W(s) \in \mathcal{D}^m$.

Disturbance decoupling implies original disturbance decoupling. However, the converse is incorrect, which can be shown by the following example.

Example

$$\begin{array}{rcl} x_1(t+1) & = & x_1(t) \vee x_2(t) \vee w(t), \\ x_2(t+1) & = & \neg w(t), \\ y(t) & = & h(x_1(t), x_2(t)) = x_1(t) \vee x_2(t), \end{array} \tag{36}$$

where x_1, x_2, w, $y \in \mathcal{D}$. When the initial state is $(x_1(0), x_2(0)) = (0,0)$, the output sequence is $y(0) = 0, y(t) = 1, t \geq 1$, for any disturbance sequence. When the initial state $(x_1(0), x_2(0))$ is nonzero, the output sequence is $y(t) = 1, t \geq 0$, no matter what the disturbance is. Thus, the original disturbance decoupling is implemented.

Example

The algebraic form of the above BN (36) is

$$\begin{aligned} x(t+1) &= Lw(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{37}$$

where $L = \delta_4[2\ 2\ 2\ 2\ 1\ 1\ 1\ 3]$, $H = \delta_2[1\ 1\ 1\ 2]$. Let $y = \delta_2^1$ (δ_2^2) represent gray (white).

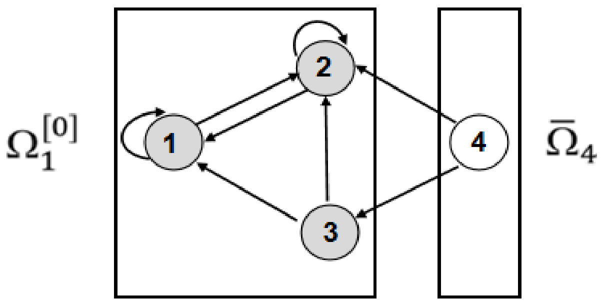


Figure 6: Vertex-colored state transition graph

Theorem

The original disturbance decoupling of BN (34) is implemented if and only if the vertex-colored state transition graph of (34) has a CP-VP.

Proof.(Sufficiency) Denote by $\mathcal{P} = \{P_i\}_{i=1}^{s}$ a CP-VP of the vertex-colored state transition graph of (34). Since $\mathcal{P}$ is perfect, for any $l_0 = 1, 2, \dots, s$, there exist l_1, l_2, $\dots$, l_t, $\dots$ such that

$$\mathcal{N}(P_{l_0}) \subset P_{l_1},\ \mathcal{N}(P_{l_1}) \subset P_{l_2}, \dots, \mathcal{N}(P_{l_{t-1}}) \subset P_{l_t}, \dots . \tag{38}$$

Since $\mathcal{P}$ is concolorous, all the vertices in each P_{l_k} correspond to the same output, i.e., for any given P_{l_k} there exists c_{l_k} such that

$$H\delta_{2^n}^{j} = \delta_{2^r}^{c_{l_k}}, \quad \forall\, j \in P_{l_k}. \tag{39}$$

Thus we conclude that the output sequence $\{\delta_{2^r}^{c_{l_i}}\}_{i=1}^{+\infty}$ is independent of the disturbance sequence.

(Necessity) It follows from algebraic form of (34) that

$$y(s)=HLw(s-1)Lw(s-2)\cdots Lw(1)Lw(0)x(0). \tag{40}$$

Since the original disturbance decoupling of BN (34) is implemented, $y(s)$ is the same for any disturbances $w(0)$, $w(1)$,...,$w(s-1)$. Thus, letting $w(t)=\delta_{2^m}^{i_t}$ for any $t\geq 0$, we can see that

$$HL_{i_{s-1}}L_{i_{s-2}}\cdots L_{i_1}L_{i_0}x(0) \tag{41}$$

is the same for any $1\leq i_1,i_2,\ldots,i_{s-1}\leq 2^m$. By (41) and the arbitrarity of $x(0)$, we have

$$HL_{i_{s-1}}L_{i_{s-2}}\cdots L_{i_1}L_{i_0}=HL_1^s \tag{42}$$

for any $1\leq i_1,i_2,\ldots,i_{s-1}\leq 2^m$ and $s\geq 0$. We write

$$\mathcal{O}_\mu := \begin{bmatrix} H \\ HL_1 \\ HL_1^2 \\ \vdots \\ HL_1^{\mu-1} \end{bmatrix} = \begin{bmatrix} \delta_{2^r}[h_{11}\ h_{12}\ \ldots\ h_{12^n}] \\ \delta_{2^r}[h_{21}\ h_{22}\ \ldots\ h_{22^n}] \\ \delta_{2^r}[h_{31}\ h_{32}\ \ldots\ h_{32^n}] \\ \vdots \\ \delta_{2^r}[h_{\mu 1}\ h_{\mu 2}\ \ldots\ h_{\mu 2^n}] \end{bmatrix}. \tag{43}$$

It follows that there exists μ^* such that

$$HL_1^{\mu} \in \{H,\ HL_1, \ldots, HL_1^{\mu^*-1}\},\ \forall\ \mu \geq \mu^*. \qquad (44)$$

The matrix $\mathcal{O}_{\mu^*}$ is just the observability matrix. Construct a vertex partition $\mathcal{P} = \{P_l\}_{l=1}^{\eta}$ of V following such a way that a and b belong to the same class of partition $\mathcal{P}$ if and only if $\text{Col}_a(\mathcal{O}_{\mu^*}) = \text{Col}_b(\mathcal{O}_{\mu^*})$. From the construction of $\mathcal{P} = \{P_l\}_{l=1}^{\eta}$, for any $a, b \in P_l$, we have $\text{Col}_a(H) = \text{Col}_b(H)$, which implies that $H\delta_{2^n}^a = H\delta_{2^n}^b$, i.e., a and b have the same color. So the vertex partition $\mathcal{P} = \{P_l\}_{l=1}^{\eta}$ is concolorous.

We claim that vertex partition $\mathcal{P} = \{P_l\}_{l=1}^{\eta}$ is also perfect, i.e., for any class P_l, its out-neighborhood $\mathcal{N}(P_l)$ is a subset of some class P_{α_l}. To prove it, for arbitrary $a, b \in P_l$, we consider all the out-neighbors of a and b. The out-neighborhoods can be written as

$$\mathcal{N}(a) = \{a_p \mid \delta_{2^n}^{a_p} = L_p \delta_{2^n}^{a}, p = 1, 2, \cdots 2^m.\}, \quad (45)$$

$$\mathcal{N}(b) = \{b_q \mid \delta_{2^n}^{b_q} = L_q \delta_{2^n}^{b}, q = 1, 2, \cdots 2^m.\}. \quad (46)$$

For any $a, b \in P_l$, it follows from the construction of $\mathcal{P}$ that

$$HL_1^t \delta_{2^n}^{a} = HL_1^t \delta_{2^n}^{b}, \ \forall\, t = 1, 2, \ldots. \quad (47)$$

Applying (42) to (47), we have

$$HL_1^{t-1} L_p \delta_{2^n}^{a} = HL_1^{t-1} L_q \delta_{2^n}^{b}, \ \forall\, t = 1, 2, \ldots. \quad (48)$$

Using (45) and (46) to (48) yields

$$HL_1^{t-1} \delta_{2^n}^{a_p} = HL_1^{t-1} \delta_{2^n}^{b_q}, \ \forall\, t = 1, 2, \ldots, \quad (49)$$

which implies that a_p and b_q are in the same class of $\mathcal{P}$.

By the arbitrariness of a and b in P_l, we conclude that there exists α_l such that $\mathcal{N}(P_l) \subset P_{\alpha_l}$, i.e., $\mathcal{P}$ is perfect.

Theorem 3.2

Assume that the VSUA stops at step ρ. Denote $\Theta = V \setminus \bigcup_{i=1}^{s_\rho} \Omega_i^{[\rho]}$. Let $\bar{\Omega}_j = \{j\}$ for any $j \in \Theta$. Construct a vertex partition as follows:

$$\mathcal{K} = \{\Omega_i^{[\rho]}, \bar{\Omega}_j |\ i \in \{1, 2, \cdots, s_\rho\}, j \in \Theta\}. \tag{50}$$

Then $\mathcal{K}$ is the finest P-VP of the vertex set V.

Proof. For any $i = 1, 2, \ldots, s_\rho$, there exists a β_i such that $\Omega_i^{[\rho]} \subset D_{\beta_i}^{[\rho]}$, which implies that

$$\mathcal{N}(\Omega_i^{[\rho]}) \subset \mathcal{N}(D_{\beta_i}^{[\rho]}) = \Omega_{\beta_i}^{[\rho]}. \tag{51}$$

Since $|\bar{\Omega}_j| = 1$ for each $j \in \Theta$, there exists γ_j such that $\bar{\Omega}_j \subset D_{\gamma_j}^{[\rho]}$, which implies that

$$\mathcal{N}(\bar{\Omega}_j) \subset \mathcal{N}(D_{\gamma_j}^{[\rho]}) = \Omega_{\gamma_j}^{[\rho]}. \tag{52}$$

From (51) and (52), we conclude that $\mathcal{K}$ is a P-VP. In the following, we show that $\mathcal{K}$ is the finest one. For any P-VP $\mathcal{P} = \{P_i\}_{i=1}^{\tau}$, Proposition 11 ensures $\Omega_i^{[\rho]} \subset P_{\alpha_i}$ for each i and some α_i. Moreover, it is obvious that $\bar{\Omega}_j \subset P_{\zeta_j}$ for each $j \in \Theta$ and some ζ_j due to $|\bar{\Omega}_j| = 1$. Thus $\mathcal{K} \sqsubset \mathcal{P}$, which implies that $\mathcal{K}$ is the finest P-VP.

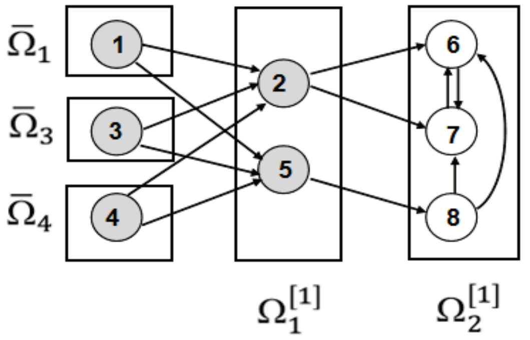
Consider the algebraic form of a BN

$$\begin{aligned} x(t+1) &= Lw(t)x(t), \\ y(t) &= Hx(t) \end{aligned} \tag{53}$$

with

$$L = \delta_8[\begin{array}{l} 2\ 6\ 2\ 2\ 8\ 7\ 6\ 6 \\ 5\ 7\ 5\ 5\ 8\ 7\ 6\ 7] \end{array}, \quad H = \delta_2[1\ 1\ 1\ 1\ 1\ 2\ 2\ 2],$$

where $x \in \Delta_8$ and $y \in \Delta_2$. Let $y = \delta_2^1$ (δ_2^2) represent gray (white). Then the vertex-colored state transition graph is



With a straightforward computation, we have $C_1 = C_3 = C_4 = \{2,5\}$, $C_2 = C_8 = \{6,7\}$, $C_5 = \{8\}$, $C_6 = \{7\}$, $C_7 = \{6\}$. Then

$$
\begin{array}{ll}
D_1^{[0]} = D_1 = \{1,3,4\}, & \Omega_1^{[0]} = \{2,5\}, \\
D_2^{[0]} = D_2 = \{2,6,7,8\}, & \Omega_2^{[0]} = \{6,7\}, \\
D_3^{[0]} = D_3 = \{5\}, & \Omega_3^{[0]} = \{8\}.
\end{array}
$$

In this example, $s_0 = 3$. Since $\Omega_1^{[0]} \cap D_2^{[0]} \neq \emptyset$ and $\Omega_1^{[0]} \cap D_3^{[0]} \neq \emptyset$, we have

$$
\begin{array}{ll}
D_1^{[1]} = \{2,6,7,8,5\}, & \Omega_1^{[1]} = \{6,7,8\}, \\
D_2^{[1]} = \{1,3,4\}, & \Omega_2^{[1]} = \{2,5\}.
\end{array}
$$

The VSUA stops at step 1. We get the finest P-VP $\mathcal{K} = \{\Omega_1^{[1]},\ \Omega_2^{[1]}, \bar{\Omega}_1,\ \bar{\Omega}_3,\ \bar{\Omega}_4\}$, where $\bar{\Omega}_1 = \{1\}, \bar{\Omega}_3 = \{3\}, \bar{\Omega}_4 = \{4\}$. Since the vertices in $\Omega_i^{[1]}$ ($i = 1,2$) have the same color, the original disturbance decoupling is implemented.

V. Conclusion

- $\bar{z} = Mx$ and $\tilde{z} = Nx$ are complementary if and only if
$$MN^{\mathrm{T}} = \mathbf{1}_{2^s, 2^{n-s}}.$$
- $\bar{z} = Mx$ is a regular basis of a regular subspace if and only if .
$$M\mathbf{1}_{2^n}^{\mathrm{T}} = 2^{n-s}\mathbf{1}_{2^s}.$$
- The BCN is decomposable with respect to inputs with order $n-s$ iff the induced digraph $\mathcal{G}$ has a PE-VP $\{S_i\}_{i=1}^{2^{n-s}}$ with $|S_i| = 2^s$.
- The disturbance decoupling is implemented iff the vertex-colored state transition graph has an ECP-VP.
- The original disturbance decoupling is implemented iff the vertex-colored state transition graph has an CP-VP.
- Vertex Set Uniting Algorithm (VSUA) is an effective algorithm.

References

- [1] Cheng, Daizhan, Qi, Hongsheng, Li, Zhiqiang. Analysis and Control of Boolean Networks. Communications & Control Engineering, Springer, 2010.
- [2] Zhu, Jiandong, and Pengjing Ju. ”Regular subspaces and invariant subspaces of Boolean control networks.” IET Control Theory and Applications 10.5 (2016): 504-508.
- [3] Zou, Yunlei, and Jiandong Zhu. ”System decomposition with respect to inputs for Boolean control networks.” Automatica 50.4 (2014): 1304-1309.
- [4] Yifeng Li, Jiandong Zhu, Bowen Li, Yang Liu, Jianquan Lu, ”A necessary and sufficient graphic condition for the original disturbance decoupling of Boolean networks.” Submitted to IEEE AC.

Many Thanks for Your Attention!

Thanks to Pengjing Ju (句鹏静), Yunlei Zou(邹云蕾), Yifeng Li(李一峰), Bowen Li(李博文), Yang Liu (刘洋), Jianquan Lu(卢剑权)