Probabilistic Boolean Networks:

Stability, Stabilization, Controllability, and Observability

Series One, Lesson Nine

Lecturer: Yuqian Guo

(School of Automation, Central South University)

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Outline

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I. Basic Concepts of PBNs

- ☞ **Boolean network (BN)** was first proposed by Kauffman[1] as a qualitative model for GRNs.

- ☞ Even though a BN provides a rougher description of GRNs, it is still capable of efficiently predicting the long-term behavior of GRNs[2].

- ☞ **Probabilistic Boolean network (PBN)**[3] is a stochastic generalization of deterministic BN, aiming to describe uncertainties and stochasticity in GRNs.

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[2] Gautier Stoll et al. "Continuous time boolean modeling for biological signaling: application of Gillespie algorithm". In: *Bmc Systems Biology* 6.1 (2012), pp. 116–116.

[3] Ilya Shmulevich, Edward R Dougherty, and Wei Zhang. "From Boolean to probabilistic Boolean networks as models of genetic regulatory networks". In: *Proceedings of the IEEE* 90.11 (2002), pp. 1778–1792.

I. Basic Concepts of PBNs

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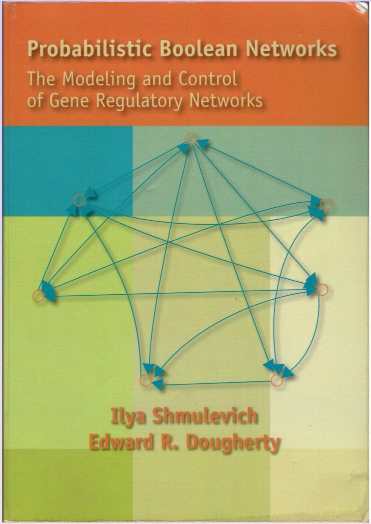

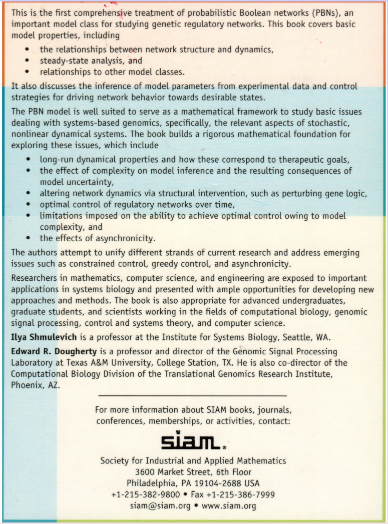

This is the first comprehensive treatment of probabilistic Boolean networks (PBNs), an important model class for studying genetic regulatory networks. This book covers basic model properties, including

- the relationships between network structure and dynamics,
- steady-state analysis, and
- relationships to other model classes.

It also discusses the inference of model parameters from experimental data and control strategies for driving network behavior towards desirable states.

The PBN model is well suited to serve as a mathematical framework to study basic issues dealing with systems-based genomics, specifically, the relevant aspects of stochastic, nonlinear dynamical systems. The book builds a rigorous mathematical foundation for exploring these issues, which include

- long-run dynamical properties and how these correspond to therapeutic goals,
- the effect of complexity on model inference and the resulting consequences of model uncertainty,
- altering network dynamics via structural intervention, such as perturbing gene logic,
- optimal control of regulatory networks over time,
- limitations imposed on the ability to achieve optimal control owing to model complexity, and
- the effects of asynchronicity.

The authors attempt to unify different strands of current research and address emerging issues such as constrained control, greedy control, and asynchronicity.

Researchers in mathematics, computer science, and engineering are exposed to important applications in systems biology and presented with ample opportunities for developing new approaches and methods. The book is also appropriate for advanced undergraduates, graduate students, and scientists working in the fields of computational biology, genomic signal processing, control and systems theory, and computer science.

Ilya Shmulevich is a professor at the Institute for Systems Biology, Seattle, WA.

Edward R. Dougherty is a professor and director of the Genomic Signal Processing Laboratory at Texas A&M University, College Station, TX. He is also co-director of the Computational Biology Division of the Translational Genomics Research Institute, Phoenix, AZ.

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Society for Industrial and Applied Mathematics
3600 Market Street, 6th Floor
Philadelphia, PA 19104-2688 USA
+1-215-382-9800 • Fax +1-215-386-7999
siam@siam.org • www.siam.org

A Reference Book

☞ This lecture is based on the work regarding stability and stabilization[4,5,6,7,8,9,10], controllability and observability[11,12,13,14].

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[4] Shiyong Zhu, Jianquan Lu, and Daniel W.C.Ho. "Finite-time Stability of Probabilistic Logical Networks: A Topological Sorting Approach". In: *IEEE Transactions on Circuits & Systems -II: Express Briefs* 67.4 (2020), pp. 695–699.

[5] Yuqian Guo et al. "Stability and Set Stability in Distribution of Probabilistic Boolean Networks". In: *IEEE Transactions on Automatic Control* 64 (2 2019), pp. 736–742.

[6] Rongpei Zhou and Yuqian Guo. "Set Stabilization in Distribution of Probabilistic Boolean Control Networks". In: *Proceedings of the 2018 13th World Congress on Intelligent Control and Automation July 4-8, 2018, Changsha, China*. 2018, pp. 274–279.

[7] Rongpei Zhou et al. "Asymptotical Feedback Set Stabilization of Probabilistic Boolean Control Networks". In: *IEEE Transactions on Neural Network & Learning Systems* (2020), DOI: 10.1109/TNNLS.2019.2955974.

[8] Liqing Wang, Yang Liu, and & Cybernetics: Systems Wu. "Stabilization and Finite-Time Stabilization of Probabilistic Boolean Control Networks". In: (2020), DOI: 10.1109/TSMC.2019.2898880.

[9] Yin Zhao and Daizhan Cheng. "On controllability and stabilizability of probabilistic Boolean control networks". In: *Science China Information Sciences* 57.1 (2014), pp. 1–14.

[10] Rui Li, Meng Yang, and Tianguang Chu. "State feedback stabilization for probabilistic Boolean networks". In: *Automatica* 50.4 (2014), pp. 1272–1278.

[11] Yin Zhao and Daizhan Cheng. "On controllability and stabilizability of probabilistic Boolean control networks". In: *Science China Information Sciences* 57.1 (2014), pp. 1–14.

[12] Fangfei Li and Jitao Sun. "Controllability of probabilistic Boolean control networks". In: *Automatica* 47.12 (2011), pp. 2765–2771.

[13] Ettore Fornasini and Maria Elena Valcher. "Observability and Reconstructibility of Probabilistic Boolean Networks". In: *IEEE Control Systems Letters* 4.2 (2020), pp. 319–324.

[14] J. Zhao and Z. Liu. "Observability of probabilistic Boolean networks". In: *Proceedings of the Chinese Control Conference, 2015*. 2015, pp. 183–186.

☞ A PBN is a randomly switched Boolean network

$$\begin{cases} x_1(t+1) = f_1^{\sigma_1(t)}\left(\left\{x_j(t) \mid j \in \mathcal{N}_1^{\sigma_1(t)}\right\}\right) \\ x_2(t+1) = f_2^{\sigma_2(t)}\left(\left\{x_j(t) \mid j \in \mathcal{N}_2^{\sigma_2(t)}\right\}\right) \\ \vdots \\ x_n(t+1) = f_n^{\sigma_n(t)}\left(\left\{x_j(t) \mid j \in \mathcal{N}_n^{\sigma_n(t)}\right\}\right) \end{cases} \tag{1}$$

- $x_i \in \mathscr{D} := \{0, 1\}$;
- $\sigma_i(t) \in \mathscr{D}_{N_i} := \{0, 1, \cdots, N_i - 1\}$, $i \in [1 : n]$, are random switching sequences; and
- f_i^j, $i \in [1 : n]$, $j \in \mathscr{D}_{N_i}$, are Boolean functions of their respective neighbouring nodes $\left\{x_k(t) \mid k \in \mathcal{N}_i^j\right\}$.

☞ Constituent networks (or contexts, subnetworks)

$$
K := \begin{array}{c}
\begin{array}{ccccc} \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{array} \\
\left[\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & N_n - 1 \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & N_n - 1 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
N_1 - 1 & N_2 - 1 & \cdots & N_{n-1} - 1 & N_n - 1
\end{array}\right]_{\left(\Pi_{i=1}^n N_i\right)\times n}
\end{array}
$$

- The jth row from bottom defines the jth subnetwork Σ_j
- There are $N := \Pi_{i=1}^n N_i$ subnetworks in total.

☞ Basic assumptions[15]:

- $\sigma_i(t)$, $i \in [1:n]$, are mutually independent;
- $\sigma_i(t)$ is independent and identically distributed,

$$\mathbb{P}\{\sigma_i(t) = j\} = \boldsymbol{p}_i^j, \quad i \in [1:n], j \in [0:N_i - 1]$$

☞ Selection probabilities of constituent networks

$$\mathbb{P}\{\sigma_1(t) = j_1, \cdots, \sigma_n(t) = j_n\} = \boldsymbol{p}_1^{j_1} \cdots \boldsymbol{p}_n^{j_n}$$

☞ Under the basic assumptions, a PBN is essentially a finite-state homogenous Markovian chain. Thus the stability of a PBN is completely determined by its state transition probabilities.

[15] A PBN does not necessarily satisfy these assumptions, such as context-sensitive PBNs and Markovian switching PBNs.

☞ The vector-form of a logic variable $\alpha \in \mathscr{D}_m$ is defined as

$$\delta_m^{m-\alpha} := \mathrm{Col}_{m-\alpha}(I_m).$$

Then,

$$f_i^j(x_1, x_2, \cdots, x_n) = L_{i,j} x_1 x_2 \cdots x_n$$

Thus, in the vector-form, the PBN becomes

$$\begin{cases} x_1(t+1) = L_1 \ltimes \sigma_1(t) \ltimes x_1(t) \ltimes x_2(t) \ltimes \cdots \ltimes x_n(t) \\ x_2(t+1) = L_2 \ltimes \sigma_2(t) \ltimes x_1(t) \ltimes x_2(t) \ltimes \cdots \ltimes x_n(t) \\ \vdots \\ x_n(t+1) = L_n \ltimes \sigma_n(t) \ltimes x_1(t) \ltimes x_2(t) \ltimes \cdots \ltimes x_n(t) \end{cases} \tag{2}$$

where

$$L_i = \left[L_{i,N_i-1}, L_{i,N_i-2}, \cdots, L_{i,1}, L_{i,0}\right]$$

☞ Define

$$x(t) = x_1(t) \ltimes x_2(t) \ltimes \cdots \ltimes x_n(t)$$

$$\sigma(t) = \sigma_1(t) \ltimes \sigma_2(t) \ltimes \cdots \ltimes \sigma_n(t)$$

The PBN becomes

$$x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes x(t)$$

where

- $\mathbf{L}$ is a logic matrix;
- $\sigma(t) \in \Delta_N$ is an independent and identically distributed (i.i.d.) random sequences with

$$\mathbb{P}\{\sigma(t) = \delta_N^j\} = \mathbb{P}\{\text{The } j\text{th subnetwork } \Sigma_j \text{ is selected}\}$$

☞ Transitional Probability Matrix (TPM) $\mathbf{P}$ [16]

$$[\mathbf{P}]_{i,j} := \mathbb{P}\left\{x(t+1) = \delta_{2^n}^{i} \;\middle|\; x(t) = \delta_{2^n}^{j}\right\}$$

$$[\mathbf{P}]_{i,j} \geq 0, \quad \sum_{i} [\mathbf{P}]_{i,j} = 1$$

Define the probability distribution vector (PDV) of σ as

$$[\boldsymbol{p}^{\sigma}]_j = \mathbb{P}\{\sigma(t) = \delta_N^j\}.$$

Then,

$$\mathbf{P} = \mathbf{L} \ltimes \boldsymbol{p}^{\sigma}$$

[16] Conventionally, the TPM is defined as $\mathbf{P}^{\top}$

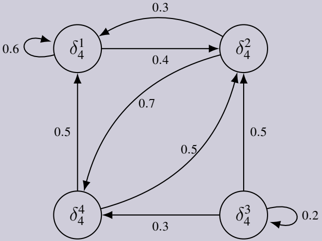
☞ State Transfer Graph (STG):

The STG of a PBN is a weighted directed graph $(\mathcal{N}, \mathcal{E}, W)$ where

- $\mathcal{N} = \Delta_{2^n}$ is the set of nodes;
- $\mathcal{E} = \left\{ (\delta_{2^n}^j, \delta_{2^n}^i) \mid [\mathbf{P}]_{i,j} > 0 \right\}$ is the set of directed edges;
- $W : \mathcal{E} \to (0, 1]$, $(\delta_{2^n}^j, \delta_{2^n}^i) \mapsto [\mathbf{P}]_{i,j}$, is the weight of edge.

Example

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.3 & 0 & 0.5 \\ 0.4 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.7 & 0.3 & 0 \end{bmatrix}$$



☞ An Example:

Consider a simplified apoptosis network [Kobayashi & Hiraish (2011)[17]]

$$\begin{cases} x_1(t+1) = f_1^{\sigma_1(t)}(x_1(t), x_2(t)) \\ x_2(t+1) = f_2^{\sigma_2(t)}(x_1(t), x_2(t), x_3(t)) \\ x_3(t+1) = f_3^{\sigma_3(t)}(x_1(t), x_2(t)) \end{cases} \tag{3}$$

- $x_1(t)$, $x_2(t)$, and $x_3(t)$ represent the concentration level of the inhibitor of apoptosis proteins, active caspase-3, and active caspase-8, respectively.
- The switching signals $\sigma_j(t) \in \{0, 1\}$, $j = 1, 2, 3$, are i.i.d. processes where

$$\begin{array}{ll} f_1^0 = x_1(t), & f_1^1 = \neg x_1(t) \wedge x_2(t), \\ f_2^0 = \neg x_1(t) \wedge x_3(t), & f_2^1 = x_1(t) \wedge x_2(t), \\ f_3^0 = x_2(t), & f_3^1 = x_1(t) \wedge x_2(t) \end{array}$$

[17] Koichi Kobayashi and Kunihiko Hiraishi. "An integer programming approach to optimal control problems in context-sensitive probabilistic Boolean networks". In: *Automatica* 47.6 (2011), pp. 1260–1264.

- Algebraic Form:

$$x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes x(t)$$

$$\mathbf{L} = [L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8]$$

$$x(t) = x_1(t) \ltimes x_2(t) \ltimes x_3(t)$$

$$\sigma(t) = \sigma_1(t) \ltimes \sigma_2(t) \ltimes \sigma_3(t)$$

$$\begin{aligned} L_1 &= \delta_8[3\ 3\ 4\ 4\ 5\ 7\ 6\ 8], & L_2 &= \delta_8[3\ 3\ 4\ 4\ 6\ 8\ 6\ 8], \\ L_3 &= \delta_8[1\ 1\ 4\ 4\ 7\ 7\ 8\ 8], & L_4 &= \delta_8[1\ 1\ 4\ 4\ 8\ 8\ 8\ 8], \\ L_5 &= \delta_8[7\ 7\ 8\ 8\ 1\ 3\ 6\ 8], & L_6 &= \delta_8[7\ 7\ 8\ 8\ 2\ 4\ 6\ 8], \\ L_7 &= \delta_8[5\ 5\ 8\ 8\ 3\ 3\ 8\ 8], & L_8 &= \delta_8[5\ 5\ 8\ 8\ 4\ 4\ 8\ 8] \end{aligned}$$

- Selection probabilities:

$$\mathbb{P}(\sigma_i(t) = j) = \boldsymbol{p}_i^j, \quad i = 1, 2, 3, \quad j = 0, 1$$

$$\begin{array}{l} \boldsymbol{p}_1^0 = 0.4, \ \ \boldsymbol{p}_1^1 = 0.6 \\ \boldsymbol{p}_2^0 = 0.7, \ \ \boldsymbol{p}_2^1 = 0.3 \\ \boldsymbol{p}_3^0 = 0.2, \ \ \boldsymbol{p}_3^1 = 0.8. \end{array}$$

For any j, decompose δ_8^j as $\delta_8^j = \delta_2^{j_1} \ltimes \delta_2^{j_2} \ltimes \delta_2^{j_3}$. Then,

$$\begin{array}{rcl} [\boldsymbol{p}^\sigma]_j & = & \mathbb{P}\{\sigma(t) = \delta_8^j\} \\ & = & \mathbb{P}\{\sigma_1(t) = \delta_2^{j_1}, \sigma_2(t) = \delta_2^{j_2}, \sigma_2(t) = \delta_2^{j_3}\} \\ & = & \boldsymbol{p}_1^{j_1}\boldsymbol{p}_2^{j_2}\boldsymbol{p}_3^{j_3} \end{array}$$

$$\Downarrow$$

$$\boldsymbol{p}^\sigma = 0.01 \times [5.6\ 22.4\ 2.4\ 9.6\ 8.4\ 33.6\ 3.6\ 14.4]^\top .$$

- The TPM:

$$
\begin{aligned}
\mathbf{P} &= \mathbf{L} \ltimes \boldsymbol{p}^{\sigma} = \sum_{j=1}^{8} [\boldsymbol{p}^{\sigma}]_j L_j \\
&= \begin{bmatrix}
0.12 & 0.12 & 0 & 0 & 0.084 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.336 & 0 & 0 & 0 \\
0.28 & 0.28 & 0 & 0 & 0.036 & 0.12 & 0 & 0 \\
0 & 0 & 0.4 & 0.4 & 0.144 & 0.48 & 0 & 0 \\
0.18 & 0.18 & 0 & 0 & 0.056 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.224 & 0 & 0.7 & 0 \\
0.42 & 0.42 & 0 & 0 & 0.024 & 0.08 & 0 & 0 \\
0 & 0 & 0.6 & 0.6 & 0.096 & 0.32 & 0.3 & 1
\end{bmatrix}.
\end{aligned}
$$

Accessibility and Communicate

- State $\delta_{2^n}^j$ is accessible from state $\delta_{2^n}^i$, denote by $i \to j$, if

$$\mathbb{P}\left\{x(t) = \delta_{2^n}^j, \text{ for some } t \geq 1 \,\middle|\, x(0) = \delta_{2^n}^i\right\} > 0$$

- Two states $\delta_{2^n}^i$ and $\delta_{2^n}^j$ that are accessible to each other are said to communicate, denote by $i \leftrightarrow j$.

Lemma

For any $i \neq j$, the following statements are equivalent:

- $\delta_{2^n}^i \to \delta_{2^n}^j$;
- $[\mathbf{P}^t]_{j,i} > 0$ *for some* t *with* $1 \leq t \leq 2^n - 1$;
- *There is a path from* $\delta_{2^n}^i$ *to* $\delta_{2^n}^j$ *in the STG.*

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☞ Recurrent States

- A state $\delta_{2^n}^j$ is said to be recurrent if

$$\mathbb{P}\left\{x(t) = \delta_{2^n}^j \text{ for some } t \geq 1 \;\middle|\; x(0) = \delta_{2^n}^j\right\} = 1.$$

Lemma

$$\delta_{2^n}^i \to \delta_{2^n}^j \text{ and } \delta_{2^n}^i \text{ is recurrent.}$$

$$\Downarrow$$

$$\delta_{2^n}^i \leftrightarrow \delta_{2^n}^j \text{ and } \delta_{2^n}^j \text{ is recurrent.}$$

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$\delta_{2^n}^i \to \delta_{2^n}^j$ and $\delta_{2^n}^i$ is recurrent.

$$\Downarrow$$

$\delta_{2^n}^i \leftrightarrow \delta_{2^n}^j$ and $\delta_{2^n}^j$ is recurrent.

☞ Invariant Set (or Closed Set)

- A subset $\mathcal{C} \subset \Delta_{2^n}$ is called an invariant subset if

$$\mathbb{P}\left\{x(t+1) \in \mathcal{C} \;\middle|\; x(t) \in \mathcal{C}\right\} = 1.$$

- A subset $\mathcal{C} \subset \Delta_{2^n}$ is invariant if and only if

$$\sum_{i \in \mathrm{idx}(\mathcal{C})} [\mathbf{P}]_{i,j} = 1 \quad \forall j \in \mathrm{idx}(\mathcal{C}) := \left\{j \;\middle|\; \delta_{2^n}^j \in \mathcal{C}\right\}$$

Lemma

The transition probability from any state to an invariant subset is increasing with time.

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☞ The Largest Invariant Subset

- The union of two invariant subsets is still invariant.
- The union of all invariant subsets contained in $\mathcal{M}$ is referred to as the **largest invariant subset** in $\mathcal{M}$, denoted by $I(\mathcal{M})$.

Proposition

Suppose that $\mathcal{M} = \{\delta_{2^n}^j \mid j \in \Lambda_0\}$, where $\Lambda_0 \subseteq [1:2^n]$. We define a sequence of subsets of indices as follows:

$$\Lambda_s = \left\{ j \in \Lambda_{s-1} \;\middle|\; \sum_{i \in \Lambda_{s-1}} [\mathbf{P}]_{i,j} = 1 \right\}, \quad s = 1, 2, \cdots.$$

Then, there must exist an integer $\mathbf{k} \leq |\mathcal{M}|$ such that $\Lambda_{\mathbf{k}} = \Lambda_{\mathbf{k}-1}$. In addition, it holds that $I(\mathcal{M}) = \{\delta_{2^n}^j \mid j \in \Lambda_{\mathbf{k}}\}$.

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☞ Probabilistic Boolean Control Network (PBCN)

$$\begin{cases} x_1(t+1) = f_1^{\sigma_1(t)}(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)) \\ x_2(t+1) = f_2^{\sigma_2(t)}(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n^{\sigma_n(t)}(u_1(t), \cdots, u_m(t), x_1(t), \cdots, x_n(t))) \end{cases}$$

$$\Downarrow$$

$$x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes u(t) \ltimes x(t)$$

TPMs

$$\mathbf{P} = \mathbf{L} \ltimes \boldsymbol{p}^{\sigma}$$

$$\mathbf{P}_j = \mathbf{L} \ltimes \boldsymbol{p}^{\sigma} \ltimes \delta_{2^m}^{j}$$

Note: $\mathbf{P}_j$ is the TPM when $u(t) \equiv \delta_{2^m}^{j}$

☞ Closed-loop TPM:

$$u(t) = Kx(t), \quad K \in \mathcal{L}_{2^m \times 2^n}$$

$$\Downarrow$$

$$\begin{aligned} x(t+1) &= \mathbf{L} \ltimes \sigma(t) \ltimes u(t) \ltimes x(t) \\ &= \mathbf{L} \ltimes \sigma(t) \ltimes K \ltimes x(t) \ltimes x(t) \\ &= \mathbf{L} \ltimes \sigma(t) \ltimes K\Phi_n x(t) \end{aligned}$$

Φ_n: Power-reducing Matrix

$$\Downarrow$$

$$\mathbf{P}_K = (\mathbf{L} \ltimes \mathbf{p}^{\sigma})K\Phi_n.$$

☞ Reachability

- x_d is said to be k-step reachable from x_0 if there is a control sequence $\mathbf{u} = \{u(t)\}$ such that

$$\mathbb{P}\{x(k; x_0, \mathbf{u}) = x_d\} > 0.$$

  x_d is said to be reachable from x_0 (denoted by $x_0 \to x_d$) if there is a control sequence $\mathbf{u} = \{u(t)\}$ such that

$$\mathbb{P}\{x(t; x_0, \mathbf{u}) = x_d \text{ for some } t \geq 1\} > 0.$$

- x_d is reachable from x_0 if and only if x_d is k-step reachable from x_0 for some $k \leq 2^n - 1$.

- Reachability Matrix

$$\mathbf{R} = \sum_{k=1}^{2^n-1} (\mathbf{P} \ltimes \mathbf{1}_{2^m})^k$$

Proposition

$\delta_{2^n}^i \to \delta_{2^n}^j \Leftrightarrow [\mathbf{R}]_{j,i} > 0$

Sketchy Proof:

$$\begin{aligned}(\mathbf{P} \ltimes \mathbf{1}_{2^m})^k &= (\mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_{2^m})^k \\ &= \sum_{\text{all possible combinations}} \mathbf{P}_{i_{k-1}} \cdots \mathbf{P}_{i_1} \mathbf{P}_{i_0}\end{aligned}$$

Thus, $\left[(\mathbf{P} \ltimes \mathbf{1}_{2^m})^k\right]_{j,i} > 0$ if and only if x_d is k-step reachable from x_0.

☞ Control Invariant Subsets

- A subset $\mathcal{C} \subseteq \Delta_{2^n}$ is called a control invariant subset if, for any state $x_0 \in \mathcal{C}$, there exists a control $u_0 \in \Delta_{2^m}$ such that

$$\mathbb{P}\{x(t+1) \in \mathcal{C} \,\big|\, x(t) = x_0\} = 1. \tag{4}$$

- The union of any two control invariant subsets is still control invariant.
- The union of all control invariant subsets contained in a given subset $\mathcal{M} \subseteq \Delta_{2^n}$ is termed as the **largest control invariant subset** contained in $\mathcal{M}$ and is denoted by $I_c(\mathcal{M})$.
- If $\mathcal{C} = \{x_e\}$ is control invariant, then, x_e is called a control fixed point.

Proposition

Suppose that $\mathcal{M}_0 = \{\delta_{2^n}^i | i \in \Lambda_0\}$. A sequence of index sets $\Lambda_s, s \in \mathbb{Z}^+$, is defined as

$$\Lambda_s = \left\{ j \in \Lambda_{s-1} \middle| \exists k \in [1:2^m], \text{s.t.} \sum_{i \in \Lambda_{s-1}} [\mathbf{P}_k]_{i,j} = 1 \right\}.$$

Subsequently, there must exist a positive integer $\eta \leqslant |\mathcal{M}_0|$ such that $\Lambda_\eta = \Lambda_{\eta+1}$. Additionally, $I_c(\mathcal{M}_0) = \{\delta_{2^n}^j | j \in \Lambda_\eta\}$ holds.

Outline

II. Stability Analysis of PBNs

☞ Consider a PBN

$$x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes x(t)$$

- $x(t) \in \Delta_{2^n}$ is the state;
- $\mathbf{L} \in \mathscr{L}_{2^n \times N2^m}$ is a logic matrix,

$$\mathbf{L} = [\mathbf{L}_1, \mathbf{L}_2, \cdots, \mathbf{L}_N];$$

- $\sigma(t) \in \Delta_N$ is an i.i.d. random sequence with a PDV $\boldsymbol{p}^\sigma$.
- The TPM is $\mathbf{P} = \mathbf{L} \ltimes \boldsymbol{p}^\sigma$.

Outline

II.1 Finite-time Stability

Definition (Finite-time Stability (FTS))

A state $x_e \in \Delta_{2^n}$ is said to be finite-time stable if there is a positive integer T such that

$$\mathbb{P}\{x(t) = x_e \mid x(0) = x_0\} = 1 \quad \forall t \geq T, \forall x_0 \in \Delta_{2^n}.$$

[Li, Yang, & Chu (2014)[a]]

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Definition (Finite-time Set Stability)

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☞ Typical Set Stability Problems:

- Synchronization of networks
- Node Synchronization
- Output Tracking

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☞ Typical Set Stability Problems:

- Synchronization of networks
- Node Synchronization
- Output Tracking

☞ Criterion of FT Stability in terms of TPM

Theorem

A PBN is finite-time stable with respect to x_e if and only if

$$\mathrm{Col}\left\{\mathbf{P}^{2^n-1}\right\} = \{x_e\} \tag{5}$$

Sketchy Proof: (Necessity) FT stability $\Rightarrow x_e$ is a fixed point, and the solution from any initial state reaches x_e with $2^n - 1$ steps. $\Rightarrow$ (5)
(Sufficiency) (5) $\Rightarrow$

$$\mathbf{P}x_e = \mathbf{P}^{2^n}x_0 = \mathbf{P}^{2^n-1}(\mathbf{P}x_0) = [x_e, \cdots, x_e](\mathbf{P}x_0) = x_e$$

$\Rightarrow x_e$ is a fixed point $\Rightarrow$ For any $t \geq 2^n$, any x_0,

$$\mathbb{P}\{x(t) = x_e \mid x(0) = x_0\} \geq \mathbb{P}\{x(2^n-1) = x_e \mid x(0) = x_0\} = 1$$

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☞ Criterion of FT Stability in terms of STG[18]

$$\mathbb{P}\{x(t) = x_e \mid x(0) = x_0\} = 1 \quad \forall t \geq T, \forall x_0 \in \Delta_{2^n}.$$

$$\Updownarrow$$

$$\begin{cases} \text{(i) } x_e \text{ is a fixed point} \\ \text{(ii) } x_0 \to x_e \quad \forall x_0 \\ \text{(iii) The paths from any } x_0 \text{ to } x_e \text{ in } \mathcal{G} \setminus (x_e, x_e) \text{ is bounded} \end{cases}$$

$$\Updownarrow$$

$$\mathcal{G} \setminus (x_e, x_e) \text{ is acyclic}$$

- **Note:** $\mathcal{G} \setminus (x_e, x_e)$ is the graph obtained from the STG $\mathcal{G}$ of the PBN by removing the self-loop of x_e

[18] Shiyong Zhu, Jianquan Lu, and Daniel W.C.Ho. "Finite-time Stability of Probabilistic Logical Networks: A Topological Sorting Approach". In: *IEEE Transactions on Circuits & Systems -II: Express Briefs* 67.4 (2020), pp. 695–699.

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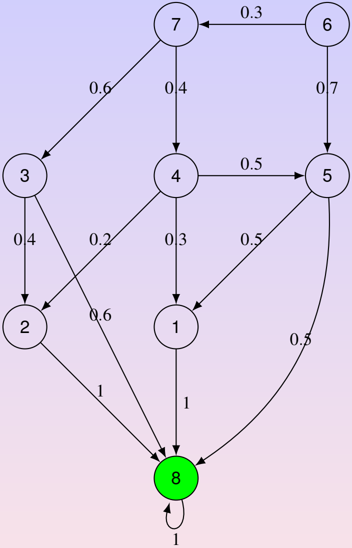
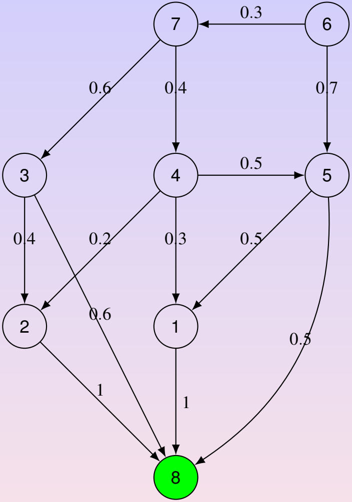


STG $\mathcal{G}$



$\mathcal{G} \setminus (x_e, x_e)$

Theorem

A PBN is finite-time stable with respect to x_e if and only if $\mathcal{G} \setminus (x_e, x_e)$ is acyclic.

[Zhu, Lu, W.C.Ho (2020)[a]]

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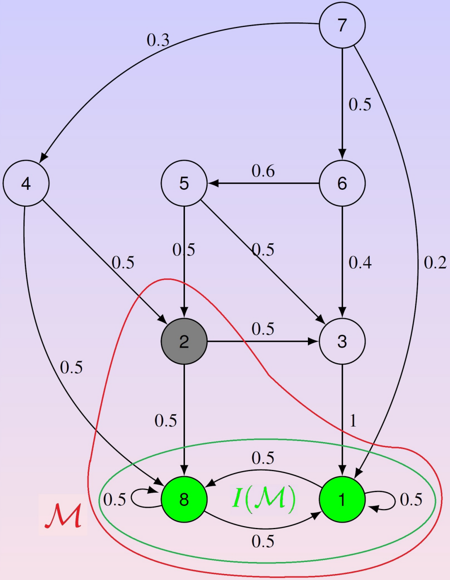
☞ Finite-time Set Stability

- Finite-time stability w.r.t. $\mathcal{M}$

  $\Leftrightarrow$ Finite-time stability w.r.t. the largest invariant subset in $\mathcal{M}$, denoted by $I(\mathcal{M})$

  $\Leftrightarrow \mathrm{Col}\{\mathbf{P}^{2^n - |I(\mathcal{M})|}\} \subseteq I(\mathcal{M})$

  $\Leftrightarrow I(\mathcal{M}) \neq \emptyset$ and the STG has no cycles outside $I(\mathcal{M})$.



The STG of a PBN that is finite-time stable w.r.t. $\mathcal{M}$

Outline

1. Basic Concepts of PBNs
2. Stability Analysis of PBNs
   - Finite-time Stability
   - Asymptotical Stability
3. Feedback Stabilization of PBNs
   - Fintie-time Feedback Stabilization
   - Asymptotical Feedback Stabilization
4. Controllability of PBNs
5. Observability of PBNs
6. Questions

II.2 Asymptotical Stability

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0.8 & 1 & 0 & 1 \end{bmatrix}$$

$$\lim_{t\to\infty} \mathbb{P}\{x(t) = \delta_4^4 \mid x(0) = x_0\}$$

$$= \lim_{t\to\infty} [\mathbf{P}^t x_0]_4 = 1$$

STG of a PBN that is not FT stable

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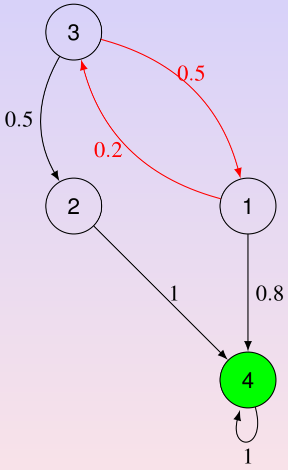

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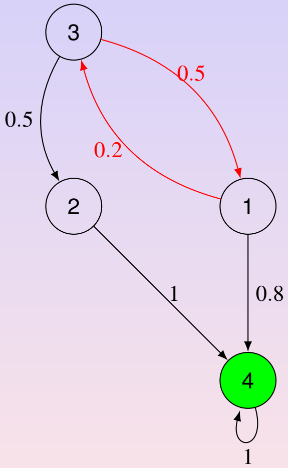

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Definition (Stability with Probability One (SPO))

A state $x_e \in \Delta_{2^n}$ is said to be stable with probability one if

$$\mathbb{P}\left\{\lim_{t\to\infty} x(t) = x_e \;\middle|\; x(0) = x_0\right\} = 1 \quad \forall x_0 \in \Delta_{2^n}.$$

[Zhao & Cheng (2014) [a]]

[a] Yin Zhao and Daizhan Cheng. "On controllability and stabilizability of probabilistic Boolean control networks". In: *Science China Information Sciences* 57.1 (2014), pp. 1–14.

Definition (Stability in Stochastic Sense (SSS))

A state $x_e \in \Delta_{2^n}$ is said to be stable in stochastic sense if

$$\lim_{t\to\infty} \mathbb{E}x(t;x_0) = x_e \quad \forall x_0 \in \Delta_{2^n}.$$

[Meng, Liu, & Feng(2017)[a]]

[a] Min Meng, Lu Liu, and Gang Feng. "Stability and l_1 gain analysis of Boolean networks with Markovian jump parameters". In: *IEEE Transactions on Automatic Control* 62.8 (2017), pp. 4222–4228.

Definition (Stability in Distribution (SD))

A state $x_e \in \Delta_{2^n}$ is said to be stable in distribution if

$$\lim_{t\to\infty} \mathbb{P}\left\{x(t) = x_e \,\middle|\, x(0) = x_0\right\} = 1 \quad \forall x_0 \in \Delta_{2^n}.$$

[Guo, Zhou, Wu, Gui, & Yang(2019)[a]]

[a] Yuqian Guo et al. “Stability and Set Stability in Distribution of Probabilistic Boolean Networks”. In: *IEEE Transactions on Automatic Control* 64 (2 2019), pp. 736–742.

- These three definitions of stability are equivalent, as shown latter.
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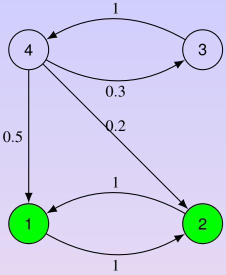

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$$\mathcal{M} = \{\delta_4^1, \delta_4^2\}$$

- The limitations

$$\lim_{t\to\infty} x(t), \quad \lim_{t\to\infty} \mathbb{E}x(t)$$

  do not exist;
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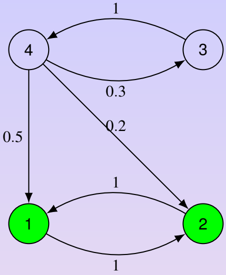

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Definition (Set Stability in Distribution (SSD))

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☞ Criterion of Stability with Probability One[19]

$$\mathbb{P}\left\{\lim_{t\to\infty} x(t) = x_e \;\middle|\; x(0) = x_0\right\} = 1 \quad \forall x_0 \in \Delta_{2^n}.$$

$$\Updownarrow$$

$$\begin{cases} x_e \text{ is a fixed point.(Thus, it is recurrent)} \\ x_0 \to x_e \quad \forall x_0.\text{(Thus, it is the unique recurrent state)} \end{cases}$$

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$$\begin{cases} x_e \text{ is a fixed point.} \\ \mathrm{Row}_i\left(\sum_{k=1}^{2^n-1} \mathbf{P}^k\right) \succ 0 \quad \text{(where } x_e = \delta_{2^n}^i\text{)} \end{cases}$$

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Theorem

A PBN is asymptotically stable w.r.t. $x_e = \delta_{2^n}^i$ *with probability one if and only if* x_e *is a fixed point and*

$$\mathrm{Row}_i\left(\sum_{k=1}^{2^n-1} \mathbf{P}^k\right) \succ 0 \tag{6}$$

[Zhao & Cheng (2014)[a]]

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Note: Condition (6) can be replaced by

$$\mathrm{Row}_i\left(\mathbf{P}^{2^n-1}\right) \succ 0$$

because the transition probability from any state to a fixed point is nondecreasing.

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☞ Criterion of asymptotical stability in distribution

- **A Necessary Condition:**

$$\lim_{t\to\infty} \mathbb{P}\left\{x(t) = x_e \;\middle|\; x(0) = x_0\right\} = 1 \quad \forall x_0 \in \Delta_{2^n}.$$

$$\Downarrow$$

$$\begin{cases} x_e \text{ is a fixed point.} \\ x_0 \to x_e \quad \forall x_0 \end{cases}$$

- It is also sufficient.

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- **It is also sufficient.**

Sketchy Proof of Sufficiency.

$$\lim_{t\to\infty} \mathbb{P}\left\{x(t) = x_e \,\middle|\, x(0) = x_0\right\} = 1 \quad \forall x_0 \in \Delta_{2^n}.$$

$$\Updownarrow$$

$$\lim_{t\to\infty} \mathbf{P}^t = \left[\begin{array}{c} 0_{(2^n-1)\times 2^n} \\ \mathbf{1}_{2^n}^T \end{array}\right] \quad \text{(Assume } x_e = \delta_{2^n}^{2^n}\text{)}$$

$$\Updownarrow$$

$$\lim_{t\to\infty} \boldsymbol{\alpha}(t) = \mathbf{1}_{2^n-1}, \quad \text{where} \quad \mathbf{P}^t := \left[\begin{array}{cc} \boldsymbol{\Gamma}^T(t) & 0_{(2^n-1)\times 1} \\ \boldsymbol{\alpha}^T(t) & 1 \end{array}\right].$$

$$\Updownarrow$$

$$\lim_{t\to\infty} \boldsymbol{\alpha}(2^n t) = \mathbf{1}_{2^n-1} \quad \text{(By Monotonicity)}$$

$$\mathbf{P}(2^n(t+1)) = \mathbf{P}(2^n t)\mathbf{P}(2^n)$$

$$\Downarrow$$

$$\boldsymbol{\alpha}(2^n(t+1)) = \boldsymbol{\Gamma}(2^n)\boldsymbol{\alpha}(2^n t) + \boldsymbol{\alpha}(2^n).$$

$$\Downarrow$$

$$\boldsymbol{\eta}(t+1) = \boldsymbol{\Gamma}(2^n)\boldsymbol{\eta}(t)$$

$$\boldsymbol{\eta}(t) := \boldsymbol{\alpha}(2^n t) - \mathbf{1}_{2^n-1}$$

$$\begin{cases} x_e \text{ is a fixed point.} \\ x_0 \to x_e \quad \forall x_0 \end{cases}$$

$$\Downarrow$$

$$\boldsymbol{\alpha}(2^n) \succ 0$$

$$\Downarrow$$

$\boldsymbol{\Gamma}(2^n)$ is strictly Schur stable

$$\Downarrow$$

$$\lim_{t\to\infty} \boldsymbol{\eta}(t) = 0$$

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Theorem

A PBN is asymptotically stable w.r.t. x_e in distribution if and only if

$$\begin{cases} x_e \text{ is a fixed point.} \\ x_0 \to x_e \quad \forall x_0. \end{cases}$$

Or, equivalently, x_e is a fixed point and

$$\mathrm{Row}_i\left(\mathbf{P}^{2^n-1}\right) \succ 0$$

[Guo, Zhou, Wu, Gui, & Yang(2019)[a]]

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☞ Criterion of asymptotical stability in stochastic sense

$$\lim_{t\to\infty} \mathbb{E}x(t;x_0) = x_e \quad \forall x_0 \in \Delta_{2^n}.$$

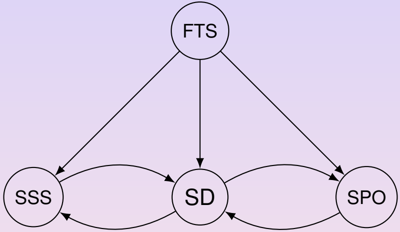
$$\Updownarrow \qquad \mathbb{E}x(t;x_0) = \mathbf{P}^t x_0$$

$$\lim_{t\to\infty} \mathbf{P}^t x_0 = x_e \quad \forall x_0 \in \Delta_{2^n}$$

$$\Updownarrow$$

Asymptotically stable in distribution

☞ **Relations between Different Definitions of Stability**



☞ Asymptotical Set Stability

$$\lim_{t\to\infty} \mathbb{P}\left\{x(t) \in \mathcal{M} \;\middle|\; x(0) = x_0\right\} = 1 \quad \forall x_0 \in \Delta_{2^n}.$$

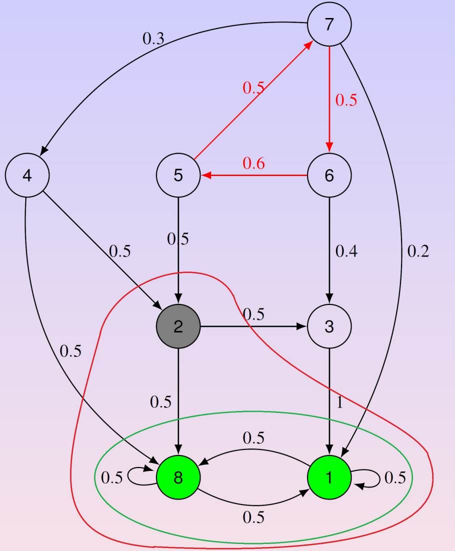
$$\Updownarrow$$

$$\lim_{t\to\infty} \mathbb{P}\left\{x(t) \in I(\mathcal{M}) \;\middle|\; x(0) = x_0\right\} = 1 \quad \forall x_0 \in \Delta_{2^n}.$$

$$\Updownarrow$$

$$\begin{cases} I(\mathcal{M}) \neq \emptyset \\ x_0 \to I(\mathcal{M}) \quad \forall x_0 \end{cases}$$

Note: $x_0 \to I(\mathcal{M})$ means $x_0 \to x$ for some $x \in I(\mathcal{M})$.



STG of a PBN that is asymptotically stable w.r.t. $\mathcal{M}$

☞ Stability of Markovian Switching PBNs

$$x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes x(t)$$

- $x(t) \in \Delta_{2^n}$
- $\mathbf{L} \in \mathscr{L}_{2^n \times n2^n}$
- $\sigma(t) \in \Delta_N$ is a homogeneous Markov chain with transition probability matrix $\mathbf{P}^\sigma$, where

$$p_{ij}^{\sigma} := [\mathbf{P}^{\sigma}]_{i,j} = \mathbb{P}\left\{\sigma(t+1) = \delta_N^i \;\middle|\; \sigma(t) = \delta_N^j\right\}.$$

- Define

$$\xi(t) := \sigma(t) \ltimes x(t) \in \Delta_{2^n N}.$$

Then, $\xi(t)$ is a homogeneous Markov chain.

- Denote the 1-step transition probability matrix of $\xi(t)$ as $\mathbf{P}^\xi$; that is,

$$p^\xi_{ij} := [\mathbf{P}^\xi]_{i,j} = \mathbb{P}\left\{\xi(t+1) = \delta^i_{2^n N} \;\middle|\; \xi(t) = \delta^j_{2^n N}\right\}.$$

Proposition

The Markovian switching PBN is finite-time (or asymptotically) $\mathcal{M}$-stable with $\mathcal{M} = \{\delta^j_{2^n} \mid j \in \Lambda_{\mathcal{M}}\}$ iff $\xi(t)$ is finite-time (or asymptotically) $\mathcal{M}_d$-stable, where

$$\mathcal{M}_d = \Delta_N \ltimes \mathcal{M} := \left\{\delta^i_N \ltimes \delta^j_{2^n} \;\middle|\; i \in [1:N],\, j \in \Lambda_{\mathcal{M}}\right\}. \quad (7)$$

- How to calculate $\mathbf{P}^\xi$?

- Define
$$\xi(t) := \sigma(t) \ltimes x(t) \in \Delta_{2^n N}.$$
Then, $\xi(t)$ is a homogeneous Markov chain.
- Denote the 1-step transition probability matrix of $\xi(t)$ as $\mathbf{P}^{\xi}$; that is,
$$p_{ij}^{\xi} := [\mathbf{P}^{\xi}]_{i,j} = \mathbb{P}\left\{\xi(t+1) = \delta_{2^n N}^{i} \mid \xi(t) = \delta_{2^n N}^{j}\right\}.$$

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- How to calculate $\mathbf{P}^\xi$?

Lemma

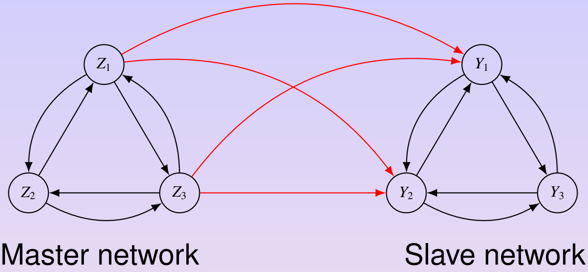
$$\mathbf{P}^{\xi} = \left(\mathbf{P}^{\sigma}\mathbf{D}_{[N,2^n]}\right) * \mathbf{L}.$$

Proof: Take any $\delta_{2^nN}^{i} = \delta_N^{i_1} \ltimes \delta_{2^n}^{i_2}$, $\delta_{2^nN}^{j} = \delta_N^{j_1} \ltimes \delta_{2^n}^{j_2}$. Then,

$$\begin{aligned}
p_{ij}^{\xi} =& \mathbb{P}(\sigma(t+1) = \delta_N^{i_1}, x(t+1) = \delta_{2^n}^{i_2} \mid \sigma(t) = \delta_N^{j_1}, x(t) = \delta_{2^n}^{j_2}) \\
=& \mathbb{P}(\sigma(t+1) = \delta_N^{i_1} \mid \sigma(t) = \delta_N^{j_1}) \\
& \times \mathbb{P}(x(t+1) = \delta_{2^n}^{i_2} \mid \sigma(t) = \delta_N^{j_1}, x(t) = \delta_{2^n}^{j_2}) \\
=& \left[\mathbf{P}^{\sigma}\delta_N^{j_1}\right]_{i_1} \left[\mathbf{L}\delta_N^{j_1}\delta_{2^n}^{j_2}\right]_{i_2} = \left[\left(\mathbf{P}^{\sigma}\delta_N^{j_1}\right)\left(\mathbf{L}\delta_N^{j_1}\delta_{2^n}^{j_2}\right)\right]_{2^n(i_1-1)+i_2} \\
=& \left[\left(\mathbf{P}^{\sigma}\mathbf{D}_{[N,2^n]}\delta_N^{j_1}\delta_{2^n}^{j_2}\right)\left(\mathbf{L}\delta_N^{j_1}\delta_{2^n}^{j_2}\right)\right]_{2^n(i_1-1)+i_2} \\
=& \left[\left(\left(\mathbf{P}^{\sigma}\mathbf{D}_{[N,2^n]}\right) * \mathbf{L}\right)\delta_N^{j_1}\delta_{2^n}^{j_2}\right]_{2^n(i_1-1)+i_2} \\
=& \left[\left(\left(\mathbf{P}^{\sigma}\mathbf{D}_{[N,2^n]}\right) * \mathbf{L}\right)\delta_{N2^n}^{2^n(j_1-1)+j_2}\right]_{2^n(i_1-1)+i_2} \\
=& \left[\left(\mathbf{P}^{\sigma}\mathbf{D}_{[N,2^n]}\right) * \mathbf{L}\right]_{i,j}
\end{aligned}$$

("$*$" represents Khatri-Rao Product of matrices.)

☞ Synchronization of PBNs



- Master Network:

$$z(t+1) = L_z z(t), \quad z(t) \in \Delta_{2^n}$$

- Slave Network:

$$y(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes z(t) \ltimes y(t), \quad z(t) \in \Delta_{2^n}$$

- $\sigma(t) \in \Delta_N$ is a homogeneous Markov chain.

Definition

- **Finite-time synchronization:**

$$\mathbb{P}\{y(t;y_0)=z(t;z_0)\}=1 \quad \forall t\geq T, \forall y_0, \forall z_0, \forall \sigma_0.$$

- **Asymptotical synchronization:**

$$\lim_{t\to\infty}\mathbb{P}\{y(t;y_0)=z(t;z_0)\}=1 \quad \forall y_0, \forall z_0, \forall \sigma_0.$$

- Rewrite the master network as

$$z(t+1) = L_z z(t) = \left[\mathbf{1}_N^T \otimes \left(L_z \mathbf{D}_{[2^n,2^n]}\right)\right] \ltimes \sigma(t) \ltimes z(t) \ltimes y(t)$$

- Define $x(t) = z(t) \ltimes y(t)$. Then, the coupled network can be expressed as

$$x(t+1) = \bar{\mathbf{L}} \ltimes \sigma(t) \ltimes x(t)$$

$$\bar{\mathbf{L}} := \left[\mathbf{1}_N^T \otimes \left(L_z \mathbf{D}_{[2^n,2^n]}\right)\right] * \mathbf{L}$$

"$*$" represents Khatri-Rao Product of matrices.

- Define $\xi(t) = \sigma(t) \ltimes x(t)$. Then, $\xi(t)$ is a homogenous Markov chain with TPM

$$\mathbf{P}^{\xi} = \left(\mathbf{P}^{\sigma} \mathbf{D}_{[N,2^{2n}]}\right) * \left[\mathbf{1}_N^T \otimes \left(L_z \mathbf{D}_{[2^n,2^n]}\right)\right] * \mathbf{L}.$$

Finite-time (or asymptotical) synchronization

$$\Updownarrow$$

$x(t) = z(t) \ltimes y(t)$ converges to $\mathcal{M}_{\mathsf{syn}}$
in Finite-time (or asymptotically)

$$\mathcal{M}_{\mathsf{syn}} = \{\delta_{2^n}^{j} \ltimes \delta_{2^n}^{j} \mid j = 1, 2, \cdots, 2^n\}$$

$$\Updownarrow$$

$\xi(t) = \sigma(t) \ltimes z(t) \ltimes y(t)$ converges to $\bar{\mathcal{M}}_{\mathsf{syn}}$
in Finite-time (or asymptotically)

$$\bar{\mathcal{M}}_{\mathsf{syn}} = \{\delta_N^i \ltimes \delta_{2^n}^{j} \ltimes \delta_{2^n}^{j} \mid i = 1, 2, \cdots, N,\ j = 1, 2, \cdots, 2^n\}$$

Example

Consider master BN

$$z(t+1) = L_z z(t), \quad z(t) \in \Delta_{2^n}$$

with $n = 2$, $L_z = \delta_4[3\ 1\ 4\ 3]$ and slave PBN

$$y(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes z(t) \ltimes y(t), \quad z(t) \in \Delta_{2^n}$$

with $\mathbf{L} = [L_1, L_2]$ and

$$L_1 = \delta_4[1\ 1\ 1\ 2\ 1\ 1\ 2\ 1\ 4\ 4\ 4\ 4\ 3\ 1\ 2\ 3],$$

$$L_2 = \delta_4[3\ 3\ 3\ 3\ 2\ 1\ 2\ 2\ 3\ 3\ 4\ 3\ 1\ 3\ 1\ 3].$$

$\sigma(t) \in \Delta_2$ is a homogenous Markov chain with

$$\mathbf{P}^\sigma = \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}.$$

- Define $\xi(t) = \sigma(t) \ltimes z(t) \ltimes y(t)$, then

$$\begin{aligned}\bar{\mathbf{L}} &:= \left[\mathbf{1}_2^T \otimes \left(L_z \mathbf{D}_{[4,4]}\right)\right] * [L_1\ L_2] \\ &= \delta_{16}[9\ 9\ 9\ 10\ 1\ 1\ 2\ 1\ 16\ 16\ 16\ 16\ 11\ 9\ 10\ 11 \\ &\qquad 11\ 11\ 11\ 11\ 2\ 1\ 2\ 2\ 15\ 15\ 16\ 15\ 9\ 11\ 9\ 11].\end{aligned}$$

$$\mathbf{P}^{\xi} = \left(\mathbf{P}^{\sigma} \mathbf{D}_{[2,16]}\right) * \bar{\mathbf{L}}$$

$$
(\mathbf{P}^{\xi})^{\top} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\
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0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

- Set of Synchronization States:

$$
\begin{aligned}
\bar{\mathcal{M}}_{\mathsf{syn}} &= \{\delta_N^i \ltimes \delta_{2^n}^j \ltimes \delta_{2^n}^j \mid i = 1, 2, \cdots, N,\ j = 1, 2, \cdots, 2^n\} \\
&= \{\delta_{16}^j \mid j \in \Lambda_0\} \\
\Lambda_0 &= \{16(i-1) + 4(j-1) + j \mid i = 1, 2,\ j = 1, 2, 3, 4\} \\
&= \{1, 6, 11, 16, 17, 22, 27, 32\}.
\end{aligned}
$$

- The largest invariant Subset in $\bar{\mathcal{M}}_{\mathsf{syn}}$

$$
I(\bar{\mathcal{M}}_{\mathsf{syn}}) = \{\delta_{16}^j \mid j \in \Lambda_2 := \{11, 16, 17, 27, 32\}\}
$$

- $$
\sum_{r \in \Lambda_2} \mathrm{Row}_r[(\mathbf{P}^{\xi})^3] > 0.
$$

  Thus, these two BNs are asymptotically synchronized.

Outline

III. Feedback Stabilization of PBNs

☞ Consider a PBCN

$$x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes u(t) \ltimes x(t)$$

- Structural matrix: $\mathbf{L} \in \mathscr{L}_{2^n \times N2^{n+m}}$;
- $\sigma(t) \in \Delta_N$ is an i.i.d. random sequence;
- TPM

$$\mathbf{P} = \mathbf{L} \ltimes \boldsymbol{p}^{\sigma} = [\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_{2^m}]$$

$$[\mathbf{P}_k]_{i,j} = \mathbb{P}\left\{x(t+1) = \delta_{2^n}^i \mid x(t) = \delta_{2^n}^j, u(t) = \delta_{2^m}^k\right\}$$

☞ Problem: Find a state-feedback

$$u(t) = Fx(k)$$

to stabilize a PBN to a point or a subset in finite-time or asymptotically.

☞ If

$$F = \delta_{2^m}[k_1, k_2, \cdots, k_{2^n}]$$

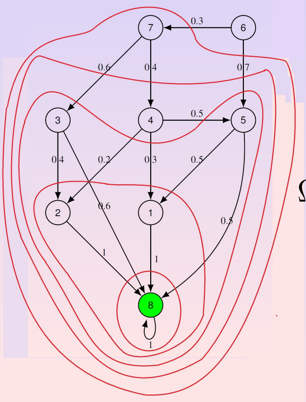
Then, the TPM of the closed loop, denoted by $\mathbf{P}_F$, is

$$\mathrm{Col}_j(\mathbf{P}_F) = \mathrm{Col}_j(\mathbf{P}_{k_j})$$

Outline

III.1 Fintie-time Feedback Stabilization

☞ Hierarchical structure of the STG of a FT stable PBN



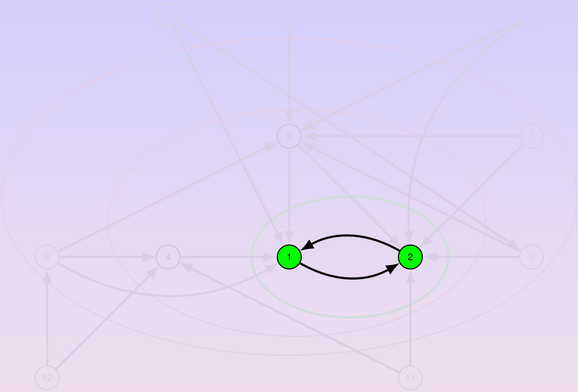
$$\Omega_0 = \{x_e\}$$

$$\Omega_1 = \left\{x \;\middle|\; \mathbb{P}\{x(t+1) = x_e \;\middle|\; x(t) = x\} = 1\right\}$$

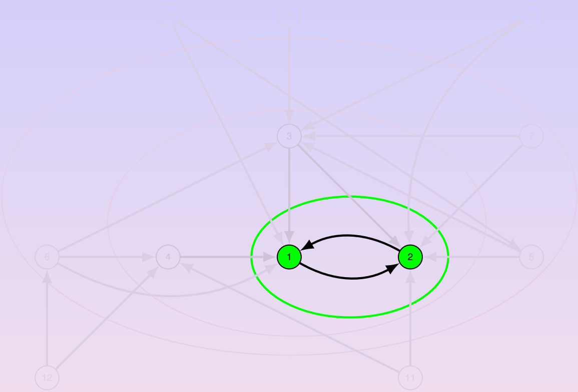
$$\Omega_k = \left\{x \;\middle|\; \mathbb{P}\{x(t+1) \in \Omega_{k-1} \;\middle|\; x(t) = x\} = 1\right\}$$

- We can always rearrange the STG into the hierarchical structure for a finite-time stable PBN.

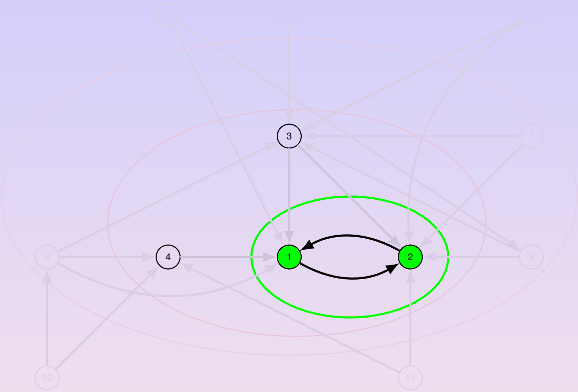
How to construct a finite-time stable STG?



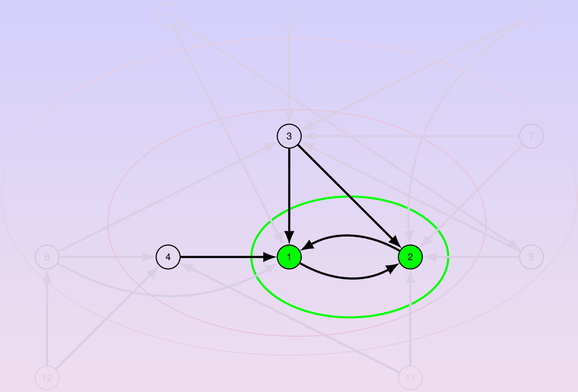
How to construct a finite-time stable STG?



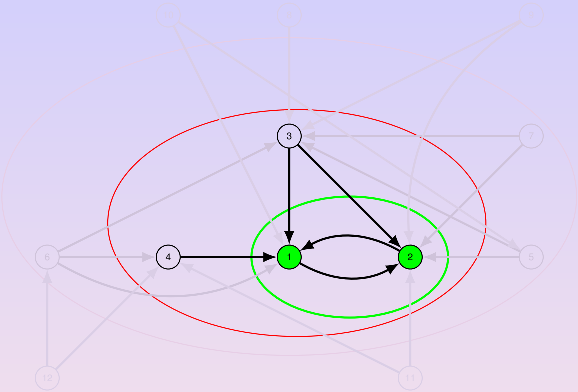
How to construct a finite-time stable STG?



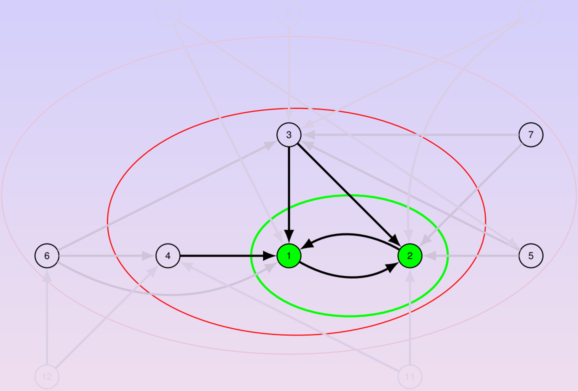
How to construct a finite-time stable STG?



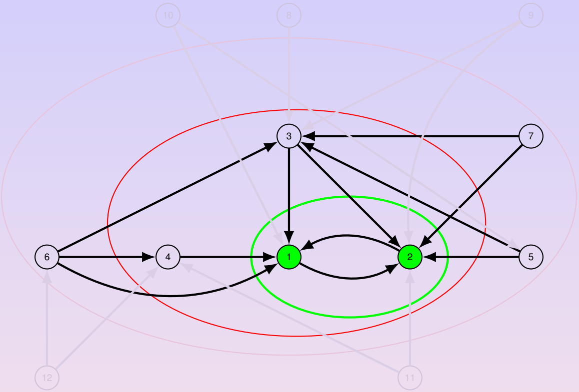
How to construct a finite-time stable STG?



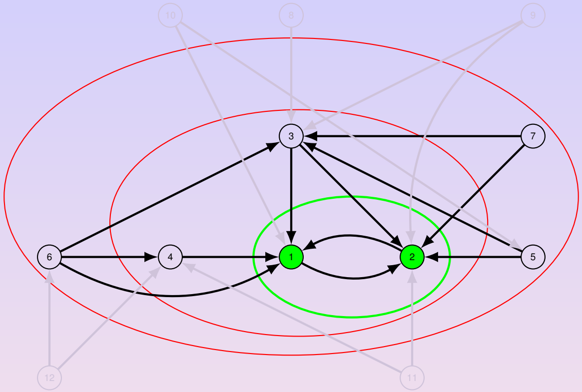
How to construct a finite-time stable STG?



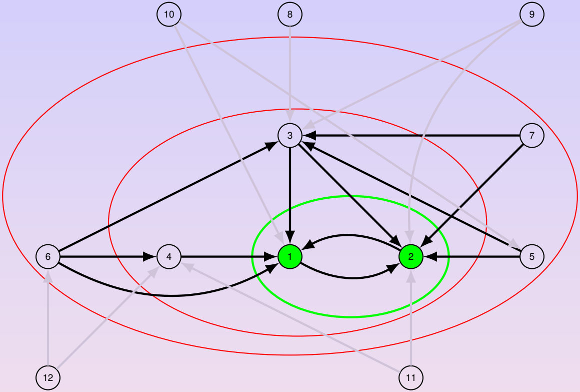
How to construct a finite-time stable STG?



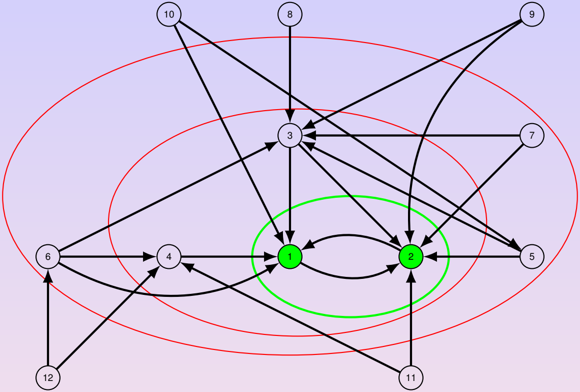
How to construct a finite-time stable STG?



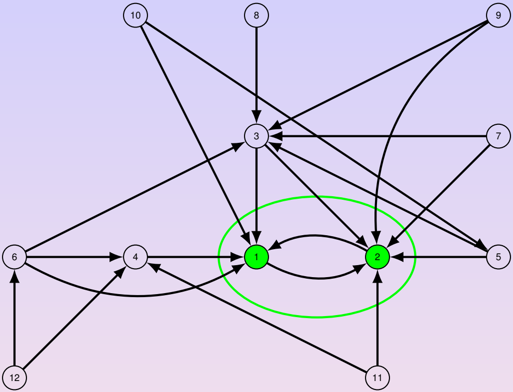
How to construct a finite-time stable STG?



How to construct a finite-time stable STG?



How to construct a finite-time stable STG?



How to construct a finite-time stable STG?

☞ Construction of Finite-time Stabilizing Feedback Based on [li, Yang, & Chu (2014)[20]]

- Define a sequence of subsets as

$$
\begin{cases}
\Omega_0 = \{x_e\} \\
\Omega_1 = \left\{x \,\middle|\, \exists u \text{ s.t. } \mathbb{P}\{x(t+1) = x_e \,\middle|\, x(t) = x, u(t) = u\} = 1\right\} \\
\Omega_k = \left\{x \,\middle|\, \exists u \text{ s.t. } \mathbb{P}\{x(t+1) \in \Omega_{k-1} \,\middle|\, x(t) = x, u(t) = u\} = 1\right\} \\
\quad k = 2, 3, \cdots
\end{cases}
$$

- If x_e is control invariant, then

$$\Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \cdots$$

[20] Rui Li, Meng Yang, and Tianguang Chu. "State feedback stabilization for probabilistic Boolean networks". In: *Automatica* 50.4 (2014), pp. 1272–1278.

Theorem

A PBN is finite-time stabilizable w.r.t. x_e by a state feedback if and only if

- x_e *is control invariant;*
- *There is a positive integer K such that $\Omega_K = \Delta_{2^n}$.*

[li, Yang, & Chu (2014)[a]]

[a] Rui Li, Meng Yang, and Tianguang Chu. "State feedback stabilization for probabilistic Boolean networks". In: *Automatica* 50.4 (2014), pp. 1272–1278.

☞ A finite-time stabilizing feedback gain F can be obtained as follows:

- Assigning a control effort $u(x_e)$ for x_e such that

$$\mathbb{P}\{x(t+1) = x_e \mid x(t) = x_e\} = 1;$$

- Assigning a control effort $u(x)$ for every $x \in \Omega_k \setminus \Omega_{k-1}$ such that

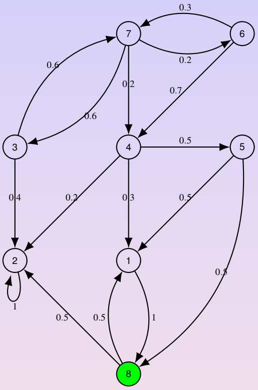
$$\mathbb{P}\{x(t+1) \in \Omega_{k-1} \mid x(t) = x\} = 1.$$

- Then,

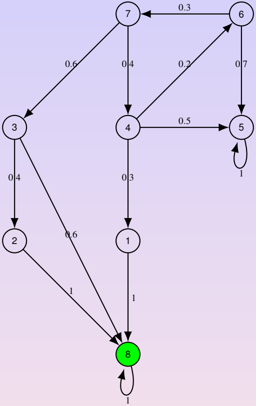
$$F = [u(\delta_{2^n}^1), u(\delta_{2^n}^2), \cdots, u(\delta_{2^n}^{2^n})]$$

[li, Yang, & Chu (2014)[21]]

[21] Rui Li, Meng Yang, and Tianguang Chu. "State feedback stabilization for probabilistic Boolean networks". In: *Automatica* 50.4 (2014), pp. 1272–1278.

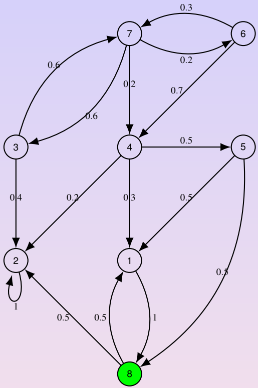


$u \equiv \delta_2^1$

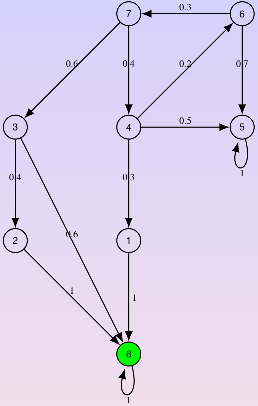


$u \equiv \delta_2^2$

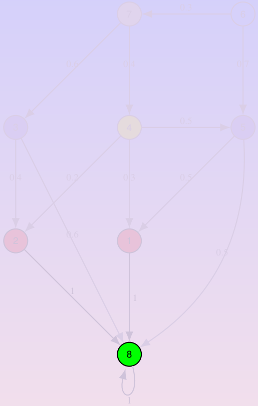
$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$



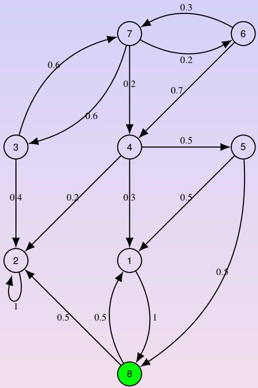
$u \equiv \delta_2^1$



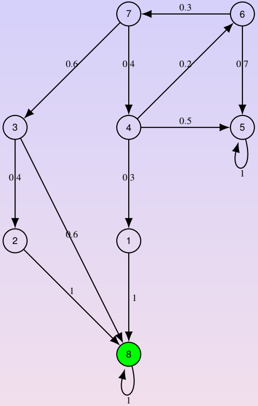
$u \equiv \delta_2^2$



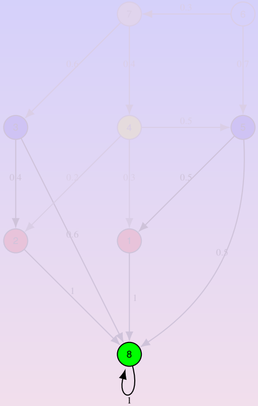
$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$



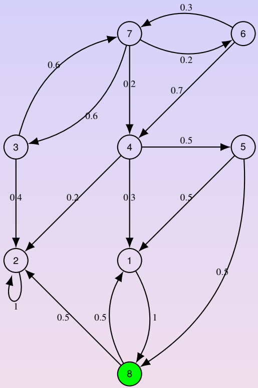
$$u \equiv \delta_2^1$$



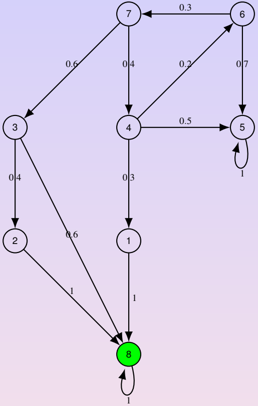
$$u \equiv \delta_2^2$$



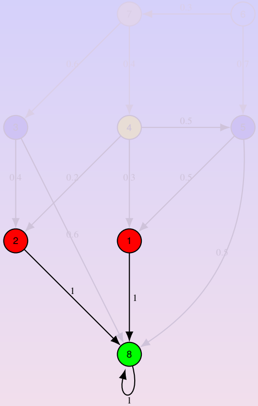
$$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$$



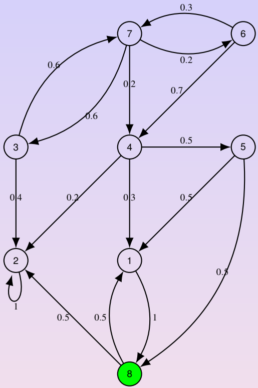
$u \equiv \delta_2^1$



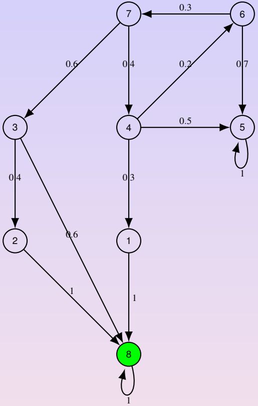
$u \equiv \delta_2^2$



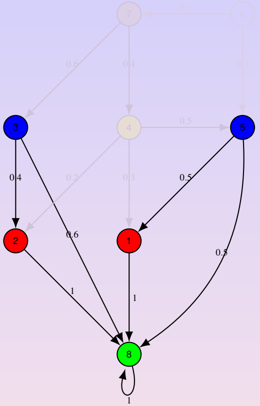
$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$



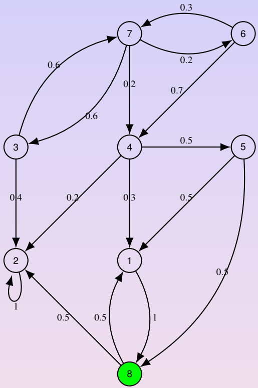
$$u \equiv \delta_2^1$$



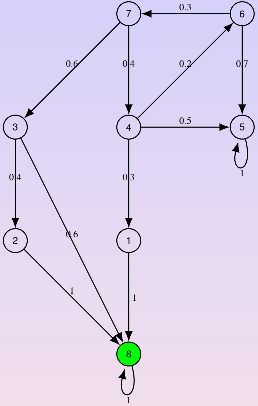
$$u \equiv \delta_2^2$$



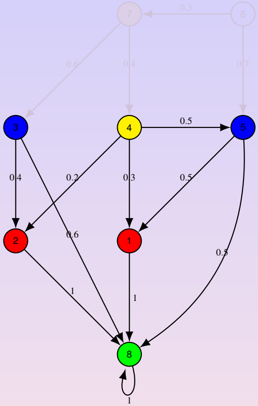
$$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$$



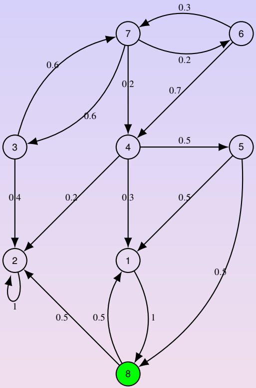
$u \equiv \delta_2^1$



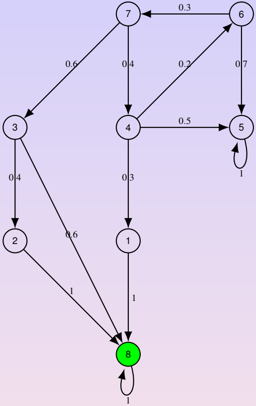
$u \equiv \delta_2^2$



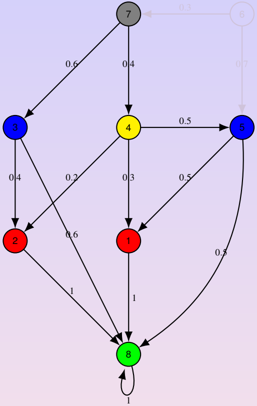
$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$



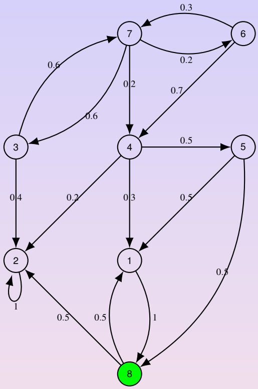
$$u \equiv \delta_2^1$$



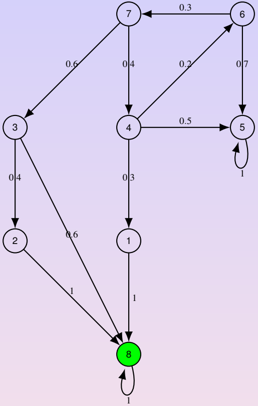
$$u \equiv \delta_2^2$$



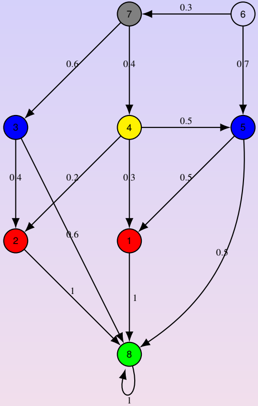
$$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$$



$u \equiv \delta_2^1$



$u \equiv \delta_2^2$



$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$

☞ Finite-time Feedback Set Stabilization

Finite-time Feedback $\mathcal{M}$-Stabilizable

$$\Updownarrow$$

Finite-time Feedback $I_c(\mathcal{M})$-Stabilizable

Outline

III.2 Asymptotical Feedback Stabilization

- Consider a PBCN

$$x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes u(t) \ltimes x(t)$$

  - Structural matrix: $\mathbf{L} \in \mathscr{L}_{2^n \times N 2^{n+m}}$;
  - $\sigma(t) \in \Delta_N$ is an i.i.d. random sequence;
  - TPM

$$\mathbf{P} = \mathbf{L} \ltimes \boldsymbol{p}^{\sigma} = [\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_{2^m}]$$

$$[\mathbf{P}_k]_{i,j} = \mathbb{P}\left\{ x(t+1) = \delta_{2^n}^i \;\middle|\; x(t) = \delta_{2^n}^j, u(t) = \delta_{2^m}^k \right\}$$

☞ Feedback Stabilizability

Theorem

A state x_e is asymptotically feedback stabilizable iff

1. x_e *is a control-fixed point, and*
2. $x_0 \to x_e \ \forall x_0$, *that is,*

$$x_e^\top \left(\mathbf{P} \ltimes \mathbf{1}_{2^m}\right)^{2^n-1} \succ 0.$$

[Zhou & Guo (2018)[a]]
[Zhou, Guo, Wu, & Gui (2019)[b]]
[Wang, Liu, Wu, Lu, & Yu (2019)[c]]

[a] Rongpei Zhou and Yuqian Guo. "Set Stabilization in Distribution of Probabilistic Boolean Control Networks". In: *Proceedings of the 2018 13th World Congress on Intelligent Control and Automation July 4-8, 2018, Changsha, China*. 2018, pp. 274–279.

[b] Rongpei Zhou et al. "Asymptotical Feedback Set Stabilization of Probabilistic Boolean Control Networks". In: *IEEE Transactions on Neural Network & Learning Systems* (2020), DOI: 10.1109/TNNLS.2019.2955974.

[c] Liqing Wang, Yang Liu, and & Cybernetics: Systems Wu. "Stabilization and Finite-Time Stabilization of Probabilistic Boolean Control Networks". In: (2020), DOI: 10.1109/TSMC.2019.2898880.

☞ Feedback Set Stabilizability

Theorem

A subset $\mathcal{M}$ is asymptotically feedback stabilizable iff

1. $I_c(\mathcal{M}) \neq \emptyset$, *and*
2. $x_0 \to I_c(\mathcal{M})\ \forall x_0$, *that is,*

$$\sum_{j \in \mathrm{idx}(I_c(\mathcal{M}))} \mathrm{Row}_j \left[(\mathbf{P} \ltimes \mathbf{1}_{2^m})^{2^n - 1} \right] \succ 0.$$

[Zhou, Guo, Wu, & Gui (2020) [a]]

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[a] Rongpei Zhou et al. "Asymptotical Feedback Set Stabilization of Probabilistic Boolean Control Networks". In: *IEEE Transactions on Neural Network & Learning Systems* (2020), DOI: 10.1109/TNNLS.2019.2955974.

☞ Design of Asymptotically Stabilizing Feedback

- Decomposition of State Space:

$$\Xi_k = \{\delta_{2^n}^s | s \in \Theta_k\}, \quad k \in [0:\lambda],$$

$$\begin{cases} \Theta_0 = \mathrm{idx}(I_c(\mathcal{M})), \\ \Theta_k = \left\{ j \in \left(\bigcup_{s=0}^{k-1} \Theta_s \right)^c \Big| \sum_{i \in \Theta_{k-1}} [\mathbf{P} \ltimes \mathbf{1}_{2^m}]_{i,j} > 0 \right\}, \\ \quad k = 1, 2, \cdots, \lambda. \end{cases} \quad (8)$$

Or equivalently,

$$\begin{cases} \Xi_0 := I_c(\mathcal{M}) \\ \Xi_k := \left\{ x \in \left(\bigcup_{s=0}^{k-1} \Xi_s \right)^c \Big| x \to \Xi_{k-1} \right\} \end{cases}$$

- For any $\delta_{2^n}^j \in \Delta_{2^n}$, we define a set of admissible controls as follows:

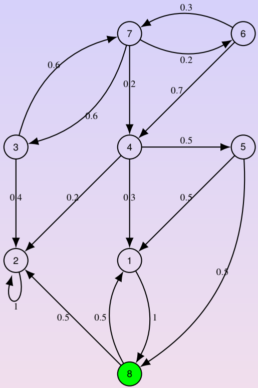
$$\kappa(\delta_{2^n}^j) := \begin{cases} \left\{ \delta_{2^m}^k \middle| \sum_{i \in \Theta_0} [\mathbf{P}_k]_{i,j} = 1 \right\}, \text{ if } \delta_{2^n}^j \in \Xi_0 \\ \left\{ \delta_{2^m}^k \middle| \sum_{i \in \Theta_{s-1}} [\mathbf{P}_k]_{i,j} > 0 \right\}, \text{ if } \delta_{2^n}^j \in \Xi_s, s \geqslant 1. \end{cases}$$

Based on the construction, under any state feedback

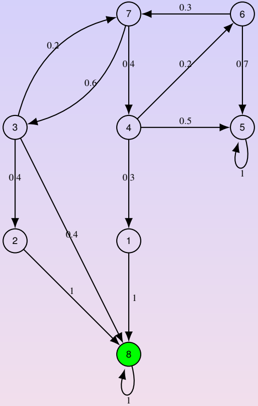
$$u(t) = Kx(t)$$

satisfying, for all $j \in [1 : 2^n]$,

$$\mathrm{Col}_j(K) \in \kappa(\delta_{2^n}^j).$$

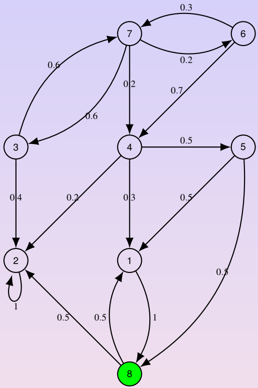


$u \equiv \delta_2^1$

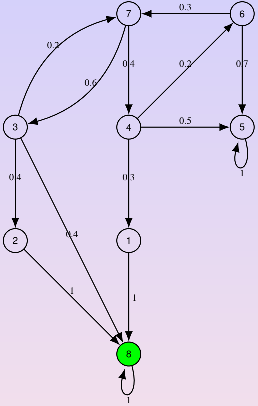


$u \equiv \delta_2^2$

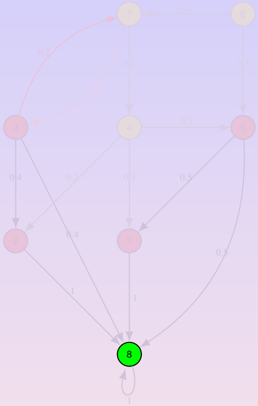
$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$



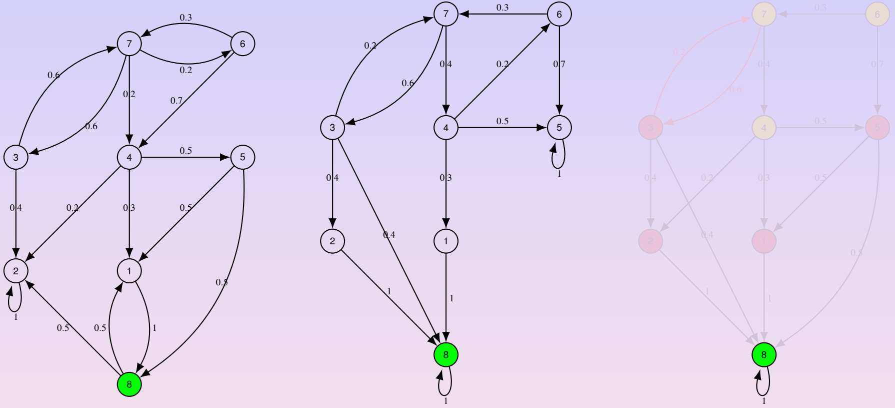
$$u \equiv \delta_2^1$$



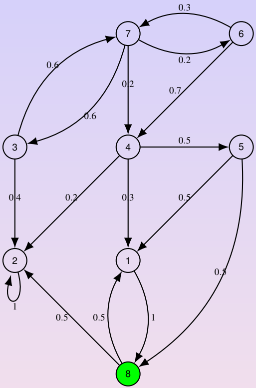
$$u \equiv \delta_2^2$$



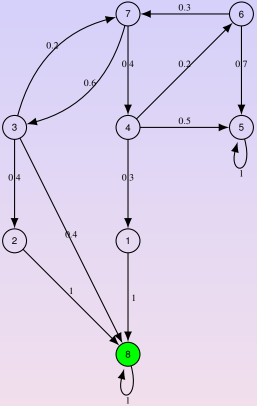
$$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$$



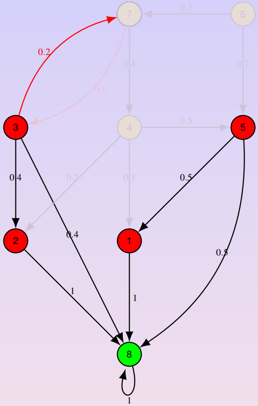
$u \equiv \delta_2^1$ $\quad$ $u \equiv \delta_2^2$ $\quad$ $u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$



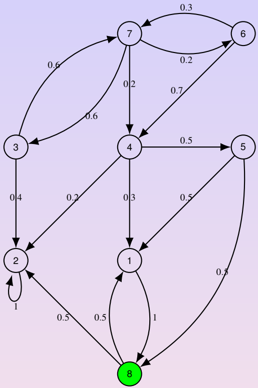
$$u \equiv \delta_2^1$$



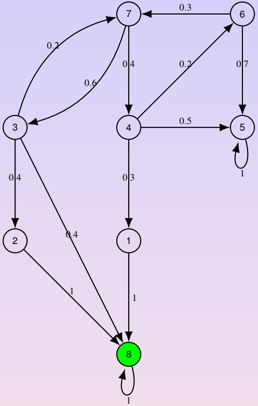
$$u \equiv \delta_2^2$$



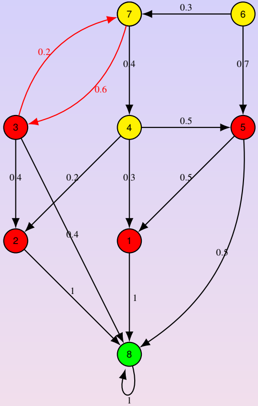
$$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$$



$u \equiv \delta_2^1$



$u \equiv \delta_2^2$



$u = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]x$

An Algorithm [Zhou, Guo, Wu, & Gui (2020)[22]]

Algorithm 1 We assume that that the PBCN (8) is asymptotically feedback $\mathcal{M}$-stabilizable in distribution. We determine the feedback gain matrix $K^* = \delta_{2^m}[r_1^* \ r_2^* \ \cdots \ r_{2^n}^*]$ that maximizes $J(\delta_{2^n}^i, \mathbf{u}), i \in [1 : 2^n]$.

Input: The one-step CD-TMP $\mathbf{P}^u = [\mathbf{P}_1^u \ \mathbf{P}_2^u \ \cdots \ \mathbf{P}_{2^m}^u]$, the $\lambda + 1$ layers $\{\Xi_k | k \in [0 : \lambda]\}$ with

$$\Xi_k = \{\delta_{2^n}^s | s \in \Theta_k\} \quad k \in [0 : \lambda]$$

Output: $\mathbf{r} = [r_1^* \ r_2^* \ \cdots \ r_{2^n}^*]$

```
r = [0 0 ··· 0] (2^n zeros),  J*(0) = 0,  η = 1
for all i ∈ Θ_0 do
    J*(δ_{2^n}^i) = 1,
        r_i* ∈ { r | Σ_{j∈Θ_0} [P_r^u]_{j,i} = 1 }
    and [r]_i = r_i*
end for
if η ⩽ λ then
    for all i ∈ Θ_η do
        J*(δ_{2^n}^i) = max_r Σ_{s∈Θ_{η−1}} J*(δ_{2^n}^s)[P_r^u]_{s,i},
        r_i* ∈ arg max_r [ Σ_{s∈Θ_{η−1}} J*(δ_{2^n}^s)[P_r^u]_{s,i} ],
    and [r]_i = r_i*
    end for
    η = η + 1
else
    break
end if
return r
```

where

$$\mathbf{r} = [\underbrace{0 \ 0 \ \cdots \ 0}_{2^n}], \quad J^*(0) = 0, \quad \eta = 1$$

$$r_i^* \in \left\{ r \,\middle|\, \sum_{j \in \Theta_0} [\mathbf{P}_r^u]_{j,i} = 1 \right\}$$

$$J^*(\delta_{2^n}^i) = \max_r \sum_{s \in \Theta_{\eta-1}} J^*(\delta_{2^n}^s)[\mathbf{P}_r^u]_{s,i},$$

$$r_i^* \in \arg\max_r \left[\sum_{s \in \Theta_{\eta-1}} J^*(\delta_{2^n}^s)[\mathbf{P}_r^u]_{s,i} \right],$$

[22] Rongpei Zhou et al. "Asymptotical Feedback Set Stabilization of Probabilistic Boolean Control Networks". In: *IEEE Transactions on Neural Network & Learning Systems* (2020), DOI: 10.1109/TNNLS.2019.2955974.

☞ **Application to Output Tracking**

We consider PBCN with q output nodes

$$\begin{cases} x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes u(t) \ltimes x(t) \\ y(t) = Hx(t) \end{cases} \tag{9}$$

where $y(t) = \ltimes_{i=1}^{q} y_i(t)$ denotes the vector form of the output variables and $H \in \mathscr{L}_{2^q \times 2^n}$.

Definition

We assume that that $y^* = \delta_{2^q}^r \in \Delta_{2^q}$ corresponds to a reference signal. The asymptotical feedback output tracking problem of the PBCN is said to be solvable if there is a state feedback $u(t) = Kx(t)$ such that, for any initial state x_0,

$$\lim_{t\to\infty} \mathbb{P}\{Hx(t) = y^*\} = 1. \tag{10}$$

Theorem

Suppose that $\sigma(t)$ is an i.i.d. process with the probability distribution vector $\mathbf{p}^{\sigma}$. We define a sequence of index sets as:

$$
\begin{aligned}
&\Lambda_0 = \{j | H\delta_{2^n}^{j} = \delta_{2^q}^{r}\},\\
&\Lambda_s = \left\{ j \in \Lambda_{s-1} \middle| \exists u = \delta_{2^n}^{k},\ \textit{s.t.,}\ \sum_{i \in \Lambda_{s-1}} [\mathbf{P}_k]_{i,j} = 1 \right\}.\\
&\quad s = 1, 2, \cdots,
\end{aligned}
$$

We denote the smallest integer by η such that $\Lambda_\eta = \Lambda_{\eta+1}$. Then, the asymptotical feedback output tracking problem is solvable iff

$$
\sum_{i \in \Lambda_\eta} \mathrm{Row}_i(\mathbf{L} \ltimes \boldsymbol{p}^{\sigma} \ltimes \mathbf{1}_{2^m})^{2^n - |\Lambda_\eta|} \succ 0. \tag{11}
$$

Outline

IV. Controllability of PBNs

☞ Consider a PBCN

$$x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes u(t) \ltimes x(t)$$

- Structural matrix: $\mathbf{L} \in \mathscr{L}_{2^n \times N2^{n+m}}$;
- $\sigma(t) \in \Delta_N$ is an i.i.d. random sequence;
- TPM

$$\mathbf{P} = \mathbf{L} \ltimes \boldsymbol{p}^{\sigma} = [\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_{2^m}]$$

$$[\mathbf{P}_k]_{i,j} = \mathbb{P}\left\{x(t+1) = \delta_{2^n}^i \mid x(t) = \delta_{2^n}^j, u(t) = \delta_{2^m}^k\right\}$$

☞ Definition of Controllability

Definition (Finite-time Controllability)

A PBCN is controllable if, for any pair of states $(\delta_{2^n}^i, \delta_{2^n}^j)$, there is a control sequence $\mathbf{u}$ and a positive integer T such that

$$\mathbb{P}\left\{x(T) = \delta_{2^n}^j \mid x(0) = \delta_{2^n}^i\right\} = 1.$$

[Li & Sun (2011)[a]]

[a] Fangfei Li and Jitao Sun. "Controllability of probabilistic Boolean control networks". In: *Automatica* 47.12 (2011), pp. 2765–2771.

Definition

A PBCN is controllable if, for any pair of states $(\delta_{2^n}^i, \delta_{2^n}^j)$, there is a control sequence $\mathbf{u}$ such that

$$\mathbb{P}\left\{x(t) = \delta_{2^n}^j \text{ for some } t \geq 1 \;\middle|\; x(0) = \delta_{2^n}^i\right\} = 1.$$

[Zhao & Cheng (2014)[a]]

[a] Yin Zhao and Daizhan Cheng. "On controllability and stabilizability of probabilistic Boolean control networks". In: *Science China Information Sciences* 57.1 (2014), pp. 1–14.

☞ Criterion for Controllability

Theorem

The PBCN is controllability iff

$$\mathbf{R} \succ 0,$$

where

$$\mathbf{R} = \sum_{k=1}^{2^n-1} \left(\mathbf{P} \ltimes \mathbf{1}_{2^m}\right)^k .$$

[Zhao & Cheng (2014)[a]]

[a] Yin Zhao and Daizhan Cheng. "On controllability and stabilizability of probabilistic Boolean control networks". In: *Science China Information Sciences* 57.1 (2014), pp. 1–14.

☞ Criterion for Finite-time Controllability

Theorem

The PBCN is finite-time controllable if and only if for any pair of states (x_0, x_d), there is a positive integer s such that

$$x_d \in \mathrm{Col}\left(\left(\mathbf{P}W_{[2^n,2^m]}\right)^s x_0\right)$$

[Li & Sun (2011)[a]]

[a] Fangfei Li and Jitao Sun. "Controllability of probabilistic Boolean control networks". In: *Automatica* 47.12 (2011), pp. 2765–2771.

Sketchy Proof:

$$x(t+1) = L_i u(t) x(t) = L_i W_{[2^n,2^m]} x(t) u(t)$$

$$\Downarrow$$

$$\mathbb{E}x(t+1) = \mathbf{P} W_{[2^n,2^m]} \mathbb{E}x(t) u(t)$$

$$\Downarrow$$

$$\begin{aligned} \mathbb{E}x(s) &= \mathbf{P} W_{[2^n,2^m]} \mathbb{E}x(s-1) u(s-1) \\ &= \cdots \\ &= \left(\mathbf{P} W_{[2^n,2^m]}\right)^s x(0) u(0) u(1) \cdots u(s-1) \end{aligned}$$

☞ Relations between Definitions

- Finite-time controllability implies controllability;
- The reverse is not true.

Example

Consider a PBCN with a state node and an input node and $\mathbf{P} = [\mathbf{P}_1 \ \mathbf{P}_2]$, where

$$\mathbf{P}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}.$$

(a) $u \equiv \delta_2^1$ (b) $u \equiv \delta_2^2$.

- This PBCN is obviously controllable because the TPM is completely connected under $u(t) \equiv \delta_2^2$.
- However, under any control sequence, the n-step TPM $\bar{\mathbf{P}}(n) \in \mathbb{R}_{2\times 2}$ is

$$\begin{aligned}\bar{\mathbf{P}}(n) =& (\mathbf{P} \ltimes u_{n-1}) \cdot \cdots \cdot (\mathbf{P} \ltimes u_1)(\mathbf{P} \ltimes u_0) \\ =& (\mathbf{P}_1)^{n-k} \cdot (\mathbf{P}_2)^k = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}^k \\ =& \begin{bmatrix} \frac{1}{3} \times (-\frac{1}{2})^k + \frac{2}{3} & \frac{2}{3} - \frac{2}{3} \times (-\frac{1}{2})^k \\ \frac{1}{3} - \frac{1}{3} \times (-\frac{1}{2})^k & \frac{2}{3} \times (-\frac{1}{2})^k + \frac{1}{3} \end{bmatrix}.\end{aligned}$$

where k the number of δ_2^2's in the control sequence. It is easy to verify the following:

$$[\bar{\mathbf{P}}(n)]_{2,1} = \frac{1}{3} - \frac{1}{3} \times (-\frac{1}{2})^k \leqslant 0.5 < 1.$$

Outline

V. Observability of PBNs

$$
\begin{cases} x(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes x(t) \\ y(t) = H \ltimes x(t), \end{cases} \tag{12}
$$

- $x(t) \in \Delta_{2^n}$, $y(t) \in \Delta_{2^q}$;
- Structural matrix: $\mathbf{L} \in \mathscr{L}_{2^n \times N2^n}$, $H \in \mathscr{L}_{2^q \times 2^n}$;
- $\sigma(t) \in \Delta_N$ is an i.i.d. random sequence with PDV $\boldsymbol{p}^\sigma$;
- TPM $\mathbf{P} = \mathbf{L} \ltimes \boldsymbol{p}^\sigma$
- For convenience, we denote the state trajectory over $[0 : \theta]$ starting from x_0 by

$$
\mathbf{x}(\theta; \sigma, x_0) := [x_0 \; x(1; \sigma, x_0) \; \cdots \; x(\theta; \sigma, x_0)]
$$

and the corresponding output trajectory by

$$
\mathbf{y}(\theta; \sigma, x_0) := [y(x_0) \; y(1; \sigma, x_0) \; \cdots \; y(\theta; \sigma, x_0)].
$$

☞ Definitions of Observability

Definition (Finite-time Observability in Probability (FTOP))

A PBN is said to be observable in probability on $[0:\theta]$ if, for any two distinct initial states $x_0, \bar{x}_0 \in \Delta_{2^n}$, it holds that

$$\mathbb{P}\left\{\mathbf{y}(\theta;\sigma,x_0) \neq \mathbf{y}(\theta;\sigma,\bar{x}_0)\right\} > 0. \tag{13}$$

A PBN is said to be finite-time observable in probability if there is a non-negative integer θ such that it is observable in probability on $[0:\theta]$.

[Zhao & Liu (2015)[a]]

[a] J. Zhao and Z. Liu. "Observability of probabilistic Boolean networks". In: *Proceedings of the Chinese Control Conference, 2015*. 2015, pp. 183–186.

Definition (Finite-time Observability with Probability One (FTOPO))

A PBN is said to be observable with probability one on $[0:\theta]$ if, for any two distinct initial states $x_0, \bar{x}_0 \in \Delta_{2^n}$, it holds that

$$\mathbb{P}\left\{\mathbf{y}(\theta;\sigma,x_0) \neq \mathbf{y}(\theta;\sigma,\bar{x}_0)\right\} = 1. \tag{14}$$

It is said to be finite-time observable with probability one if a non-negative integer θ exists, such that the PBN is observable with probability one on $[0:\theta]$.
[Zhou, Guo, & Gui (2019)[a]]

[a] Rongpei Zhou and Yuqian Guo. "Set Stabilization in Distribution of Probabilistic Boolean Control Networks". In: *Proceedings of the 2018 13th World Congress on Intelligent Control and Automation July 4-8, 2018, Changsha, China*. 2018, pp. 274–279.

Definition (Asymptotical Observability in Distribution (AOD))

A PBN is said to be asymptotically observable in distribution if, for any two distinct initial states $x_0, \bar{x}_0 \in \Delta_{2^n}$, it holds that

$$\lim_{t\to\infty} \mathbb{P}\left\{\mathbf{y}(t;\sigma,x_0) \neq \mathbf{y}(t;\sigma,\bar{x}_0)\right\} = 1. \tag{15}$$

[Zhou, Guo, & Gui (2019)[a]]

[a] Rongpei Zhou and Yuqian Guo. "Set Stabilization in Distribution of Probabilistic Boolean Control Networks". In: *Proceedings of the 2018 13th World Congress on Intelligent Control and Automation July 4-8, 2018, Changsha, China*. 2018, pp. 274–279.

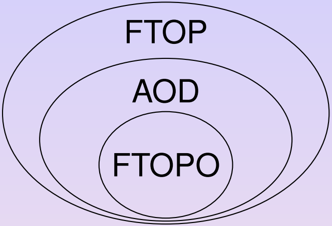


Figure 1: The ellipses labeled with FTOP, AOD, and FTOPO represent the set of PBNs that are observable in the sense of FTOP, AOD, and FTOPO, respectively.

Definition

Given a time instant $T \in \mathbb{Z}_+$, a PBN is observable in $[0, T]$ if, for every admissible output sequence $\mathbf{y}(0), \mathbf{y}(1), \ldots, \mathbf{y}(T)$, it is possible to uniquely identify the corresponding initial condition $\mathbf{x}(0)$. The PBN is observable if it is observable in some interval $[0, T]$

[Fornasini & Valcher (2020)[a]]

[a] Ettore Fornasini and Maria Elena Valcher. "Observability and Reconstructibility of Probabilistic Boolean Networks". In: *IEEE Control Systems Letters* 4.2 (2020), pp. 319–324.

Definition (Weakly Reconstructibility)

Given a PBN and a time instant $T \in \mathbb{Z}_+$, the PBN is weakly reconstructible in $[0, T]$ if for every admissible output sequence $\mathbf{y}(0), \mathbf{y}(1), \ldots, \mathbf{y}(T)$, there exists $\tau \in [0, T]$ (depending on the specific output sequence) such that the knowledge of the output samples $\mathbf{y}(0), \mathbf{y}(1), \ldots, \mathbf{y}(\tau)$ allows to uniquely identify $\mathbf{x}(\tau) \in \mathcal{L}_N$. The PBN is weakly reconstructible if it is weakly reconstructible in some interval $[0, T]$.

[Fornasini & Valcher (2020)[a]]

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[a] Ettore Fornasini and Maria Elena Valcher. "Observability and Reconstructibility of Probabilistic Boolean Networks". In: *IEEE Control Systems Letters* 4.2 (2020), pp. 319–324.

Definition (Strongly Reconstructibility))

Given a PBN and a time instant $T \in \mathbb{Z}_+$, we say that the PBN is strongly reconstructible in $[0, T]$ if, given any admissible output sequence $\mathbf{y}(0), \mathbf{y}(1), \ldots, \mathbf{y}(T) \in \mathcal{L}_P$, it is possible to uniquely identify $\mathbf{x}(T) \in \mathcal{L}_N$. The PBN (3) is strongly reconstructible if it is strongly reconstructible in some interval $[0, T]$.

[Fornasini & Valcher (2020)[a]]

[a] Ettore Fornasini and Maria Elena Valcher. "Observability and Reconstructibility of Probabilistic Boolean Networks". In: *IEEE Control Systems Letters* 4.2 (2020), pp. 319–324.

☞ [Fornasini & Valcher (2020)[23]]

Observability Matrix

$$\mathcal{O}_{\sigma,T} := \begin{bmatrix} H \\ HL_{\sigma(0)} \\ HL_{\sigma(1)}L_{\sigma(0)} \\ \vdots \\ HL_{\sigma(T-1)} \dots L_{\sigma(1)}L_{\sigma(0)} \end{bmatrix}.$$

Proposition 1: Given a PBN (3) and a time instant $T \in \mathbb{Z}_+$, let $\sigma_1, \sigma_2, \dots, \sigma_R$, where $R = M^T$, be all the possible distinct sequences $\sigma(t), t \in [0, T-1]$, taking values in $[0, M]$. The PBN is observable in $[0, T]$ if and only if for every choice of the indices $i, j \in [1, R]$ and $h, k \in [1, N]$, condition $\mathcal{O}_{\sigma_i,T}\ \delta_N^h = \mathcal{O}_{\sigma_j,T}\ \delta_N^k$ implies $h = k$.

If all columns of the matrix

$$\mathcal{O}_T := \begin{bmatrix} \mathcal{O}_{\sigma_1,T} & \mathcal{O}_{\sigma_2,T} & \cdots & \mathcal{O}_{\sigma_R,T} \end{bmatrix} \tag{6}$$

are distinct, then the PBN is observable in $[0, T]$ (and, in this particular case, one can identify from the output observation also the sequence σ_i).

[23] Ettore Fornasini and Maria Elena Valcher. "Observability and Reconstructibility of Probabilistic Boolean Networks". In: *IEEE Control Systems Letters* 4.2 (2020), pp. 319–324.

☞ Observability Analysis Based on Set Reachability

Definition (Finite-time Reachability)

Assume that $\mathcal{M}_0,\ \mathcal{M}_d \subset \Delta_{2^n}$ are the initial and target subsets, respectively. $\mathcal{M}_d$ is said to be reachable with probability one from $\mathcal{M}_0$ on $[0:\theta]$ if, for any initial state $x_0 \in \mathcal{M}_0$, it holds that

$$\mathbb{P}\left\{\exists k \in [0:\theta],\ s.t.\ x(k;\sigma,x_0) \in \mathcal{M}_d\right\} = 1. \tag{16}$$

$\mathcal{M}_d$ is said to be finite-time reachable with probability one from $\mathcal{M}_0$ if there is a θ such that $\mathcal{M}_d$ is reachable with probability one from $\mathcal{M}_0$ on $[0:\theta]$.

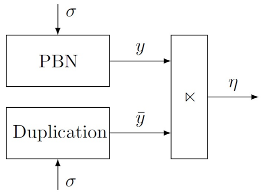
Definition (Asymptotical Reachability)

A target subset $\mathcal{M}_d \subset \Delta_{2^n}$ is said to be asymptotically reachable in distribution from an initial subset $\mathcal{M}_0 \subset \Delta_{2^n}$ if, for any initial state $x_0 \in \mathcal{M}_0$, it holds that

$$\lim_{t\to\infty} \mathbb{P}\left\{\exists k \in [0:t],\ s.t.\ x(k;\sigma,x_0) \in \mathcal{M}_d\right\} = 1. \tag{17}$$

- **Parallel Extension:**

$$\begin{cases} \bar{x}(t+1) = \mathbf{L} \ltimes \sigma(t) \ltimes \bar{x}(t) \\ \bar{y}(t) = H\bar{x}(t) \end{cases}$$



- **Interconnected Network:**
Define $\xi(t) := x(t) \ltimes \bar{x}(t)$, $\eta(t) := y(t) \ltimes \bar{y}(t)$, $\sigma_\xi(t) := \sigma(t)$. Then,

$$\begin{cases} \xi(t+1) = \mathbf{L}_\xi \sigma_\xi(t) \xi(t) \\ \eta(t) = H_\xi \xi(t), \end{cases}$$

$$\mathbf{L}_\xi = \mathbf{L} W_{[2^n, N \cdot 2^n]} \mathbf{L} W_{[N \cdot 2^n, N \cdot 2^n]} W_{[N, N \cdot 2^n]} M_{r,N}, \tag{18}$$

$$H_\xi = H W_{[2^q, 2^n]} H W_{[2^n, 2^n]}, \tag{19}$$

- **Distinguishable Set:**

$$\mathcal{G}_H = \{x \ltimes \bar{x} | Hx \neq H\bar{x}\}.$$

- **Initially Indistinguishable Set:**

$$\mathcal{G}_{I\backslash H} := \left\{ \delta_{2^n}^i \ltimes \delta_{2^n}^j \middle| \mathrm{Col}_i(H) = \mathrm{Col}_j(H) \text{ and } i < j \right\}.$$

Proposition

(a) *The original PBN is finite-time observable with probability one iff the subset $\mathcal{G}_H$ is finite-time reachable with probability one from $\mathcal{G}_{I\setminus H}$ for the interconnected PBN.*

(b) *The original PBN is asymptotically observable in distribution iff the subset $\mathcal{G}_H$ is asymptotically reachable in distribution from $\mathcal{G}_{I\setminus H}$ for the interconnected PBN.*

Outline

Questions

- ☞ Are there other reasonable ways to define controllability and observability of PBNs?
- ☞ If your answer is "yes", what is the difference between the existing definitions and yours?
- ☞ If one of your definitions tells a different story, find a sufficient or even a necessary and sufficient condition.
- ☞ Now you are ready to write a paper and publish it.

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Thank you!