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随机系统分析与设计中的算子谱技术和 H-表示技术

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汇报内容主要来自于下列2篇论文

- W. Zhang and B. S. Chen, On stabilizability and exact observability of stochastic systems with their applications, Automatica, 40(1), pp.87-94, Jan. 2004

- W. Zhang, B. S. Chen. H-Representation and Applications to Generalized Lyapunov Equations and Linear Stochastic Systems, IEEE Trans. Automatic Control, 57(12): pp. 3009-3022, 2012.

汇报提纲

一. 经典结论的回顾：一些启发性的例子

二. 算子谱技术及其应用

三. H-表示技术及其应用

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第一部分

经典结论的回顾：一些启发性的例子

一. 经典结论的回顾

熟知对于下列 n-维时不变线性常微分方程

$$\dot{x}(t) = Ax(t),\ x(0) = x_0 \tag{1.1}$$

我们有下列经典的结果:

定理1.1（Hurwitz 定理）. 系统（1)是渐近稳定的充分必要条件是矩阵 A 的所有特征值或谱都在开的左半复平面内，即

$$\lim_{t\to\infty} x(t) = 0 \Leftrightarrow \sigma(A) \subset \mathbb{C}^-$$

若取 Lyapunov 函数 $V(x) = x'Px$, 则用Lyapunov稳定性理论，亦有下列结论:

定理1.2（Lyapunov 定理）. 系统（1)是渐近稳定的充分必要条件是下列 Lyapunov 方程

$$PA + A'P = -Q,\ Q > 0$$

或 Lyapunov 不等式

$$PA + A'P < 0$$

有正定矩阵解 $P > 0$.

评注1.1. 尽管定理1.1 和定理1.2 都是系统(1)渐近稳定性的充分必要刻画，但是又各有优缺点，从而在现代控制理论中引领出不同的研究方向。 具体而言，

- 从对线性时不变系统的稳定性刻画而言， 定理1.1 更加准确有效，它不仅告诉我们系统 (1.1)是否渐近稳定，而且通过 A 的特征值在左半复平面内的分布情况，我们大体可以判断系统(1)的收敛速度。 而定理 1.2 只能告诉我们系统是否渐近稳定。
- 当然， 用 Lyapunov 函数方法的更大优势在于处理非线性系统或时变系统，此时特征值方法失效。 而且即使对于线性时变系统

$$\dot{x}(t) = A(t)x(t),\ x(0) = x_0$$

特征值方法也是失效的。

$$u(t) = Kx(t)$$

- 定理 1.1 是现代控制理论极点配置这一方向的基础。 所谓极点配置是寻求定常反馈增益矩阵K， 使闭环系统的矩阵 $(A+BK)$ 的特征值正好是复平面指定的 n 个点 $\lambda_1,\lambda_2,\cdots,\lambda_n$, 即 $\sigma(A+BK)=\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$.

理论上亦可证明：矩阵 A 的特征值在左半复平面离虚轴越远，系统（1.1）的收敛速度越快，越是靠近虚轴收敛速度越慢。

例 1.1. 考虑下列 x_1 -系统和 x_2 -系统：

$$\dot{x}_1(t)=\begin{bmatrix}-0.01 & 0\\ 0 & -0.2\end{bmatrix}x_1(t),\quad x_1(0)=\begin{bmatrix}0.5\\0.3\end{bmatrix}$$

$$\dot{x}_2(t)=\begin{bmatrix}-5 & 0\\ 0 & -6\end{bmatrix}x_2(t),\quad x_2(0)=\begin{bmatrix}0.5\\0.3\end{bmatrix}$$

显然，x_1-系统的特征值更靠近虚轴，因而收敛速度更慢。 Fig.1 和 Fig.2 的系统状态响应曲线也证明了这个事实。

评注 1.2 .对于受控系统 $\dot{x}(t)=Ax(t)+Bu(t),\quad x(0)=x_0$，为了得到理想的收敛速度，应该根据工程需要进行极点配置。

一、经典结论的回顾

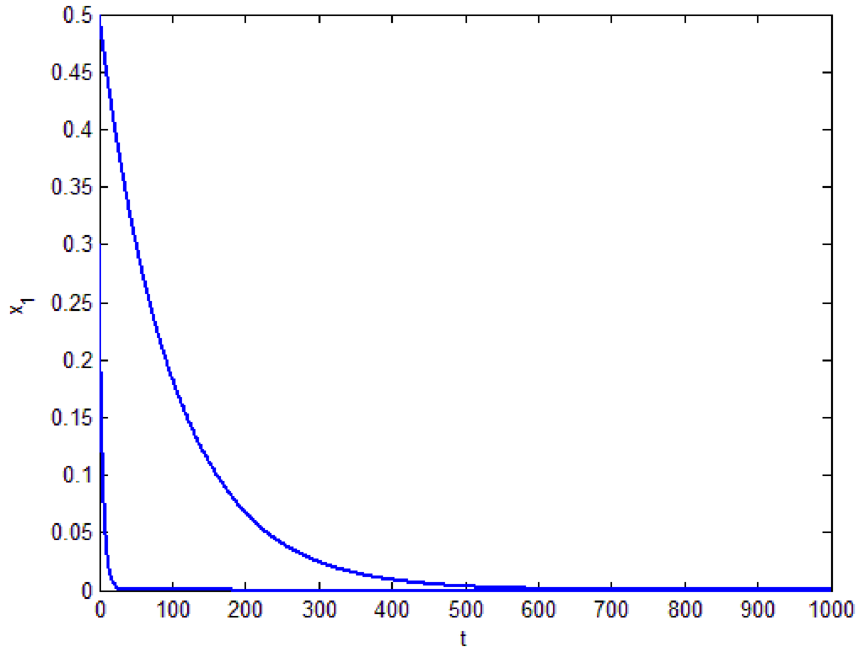


Fig.1 x_1-系统状态响应曲线

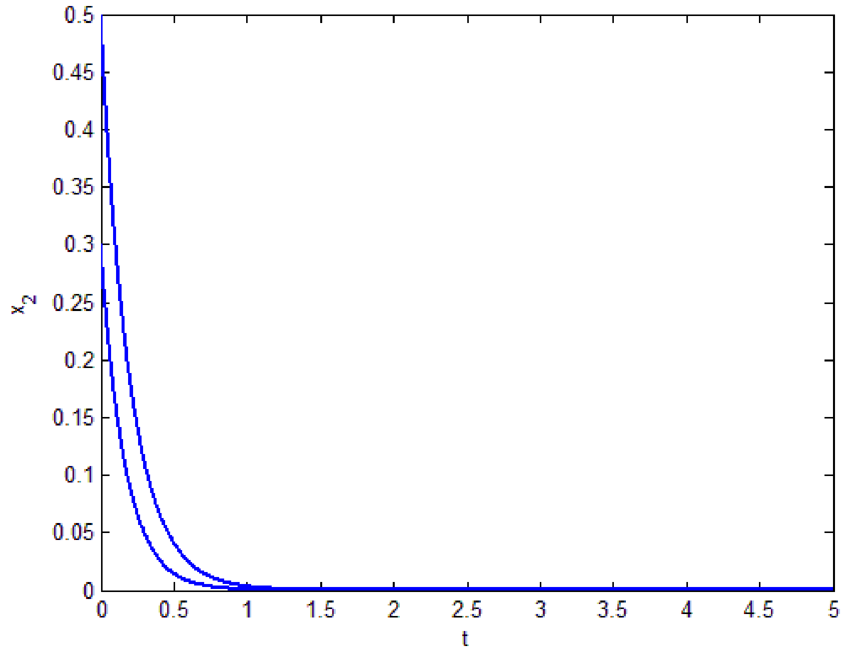
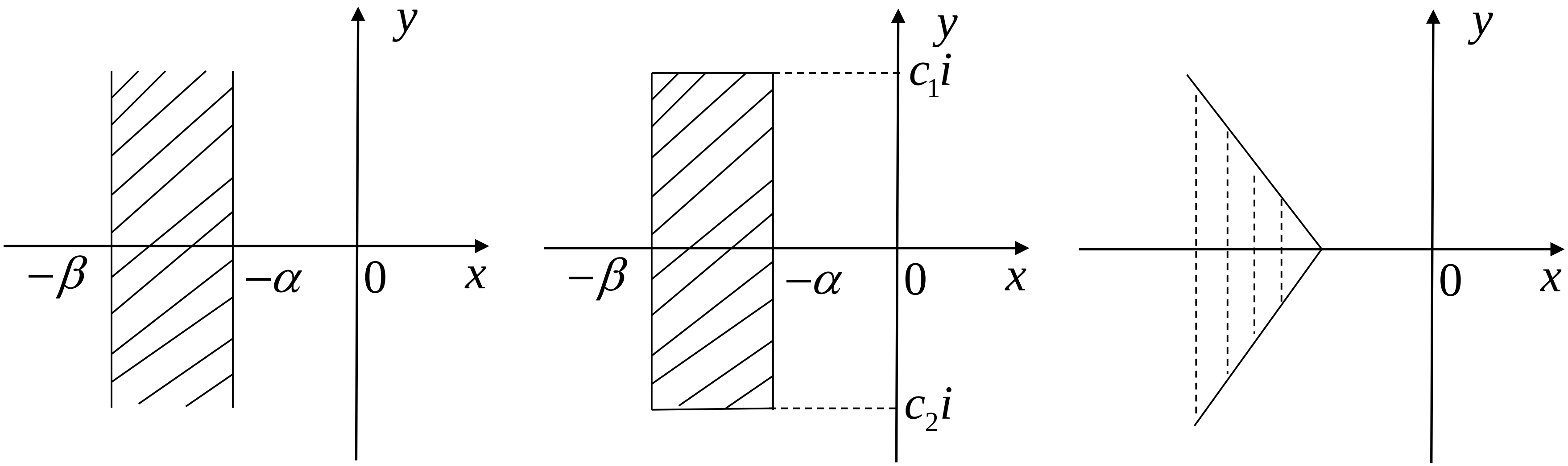


Fig.2 x_2-系统状态响应曲线

工程上几类主要的极点配置区域：



带形区域　　　矩形区域　　　扇形区域

- M. Chilali and P. Gahinet, “ H_∞ design with pole placement constraints:An LMI approach, IEEE Trans. Autom. Control, vol. 41, no.3, pp. 358–367, Mar. 1996.
- M. Chilali, P. Gahinet, and P. Apkarian, “Robust pole placementin LMI regions,” IEEE Trans. Autom. Control, vol. 44, no. 12, pp.2257–2269, Dec. 1999.

定理 1.3（T. Kailath, Linear Systems, 1998）. 考虑下列状态-输出系统

$$\begin{cases} \dot{x}(t) = Ax(t), x(0) = x_0 \\ y(t) = Dx(t) \end{cases} \tag{1.2}$$

(A,H) 是完全能观的充分必要条件是不存在非零复向量 $\xi \in \mathbb{C}^n$，复数 $\lambda \in \mathbb{C}$，使下列方程组成立：

$$A\xi = \lambda\xi, D\xi = 0, \lambda \in \mathbb{C}.$$

(A,H) 是完全能检测的充分必要条件是不存在复向量 $\xi \in \mathbb{C}^n$，复数 $\mathrm{Re}(\lambda) \geq 0$，使下列方程组成立：

$$A\xi = \lambda\xi, D\xi = 0, \mathrm{Re}(\lambda) \geq 0.$$

评注3： 定理 1.3 是完全能观和能检测的特征值判据，其他判据见现代控制理论的标准教科书。能检测性是较能观性更弱的概念。

一. 经典结论的回顾

问题 1： 对于下列线性随机时不变 Ito 系统，如何建立与定理1 相对应的特征值判据或谱判据？（在均方稳定性意义下）

$$dx(t) = Ax(t)dt + Cx(t)dW(t),\ x(0) = x_0 \tag{1.3}$$

其中$W(t)$是标准 Wiener 过程。

问题 2： 对下列随机状态-输出方程

$$\begin{cases} dx(t) = Ax(t)\mathrm{d}t + \mathrm{C}\,x(\mathrm{t})dW(\mathrm{t}),\ x(0) = x_0 \\ y(t) = Dx(t) \end{cases} \tag{1.4}$$

其精确能观性和能检测性的Hautus-型判据是什么？

问题 3： 如何恰当地定义随机控制系统的极点配置（pole placement）？

$$dx(t) = [Ax(t) + \mathrm{Bu(t)}]dt + [Cx(t) + \mathrm{Du(t)}]dW(t),\ x(0) = x_0 \tag{1.5}$$

第二部分

算子谱技术及其应用

2.1 均方稳定性的谱刻画

定义 2.1. 线性时不变系统

$$dx(t) = Ax(t)dt + Cx(t)dW(t),\ x(0) = x_0$$

是渐近均方稳定的或简称(A, C)是稳定的，意指

$$\lim_{t\to\infty} E\|x(t)\|^2 = 0$$

定义 2.2. （**Zhang and Chen, Automatica, 2004)** 针对上述系统，定义广义 **Lyapunov** 算子如下($\frac{n(n+1)}{2}$ – 维对称算子)：

$$L_{A,C} : X \in S^n \mapsto AX + XA' + CXC' \in S^n$$

其中， S^n 是对称矩阵集合。

定理 2.1. （**A,C**）是稳定的充分必要条件是 $\sigma(L_{A,C}) \subset \mathbb{C}^-$.

利用广义 **Lyapunov** 算子 $L_{A,C}$ ，可以引进新的稳定性定义：

定义 **2.3**. （**A,C**）是临界稳定的，如果 $\sigma(L_{A,C}) \subset \mathbb{C}^{-,0}$ **(**闭的左半复平面）**.**

评注 **2.1**. 通过引进广义 **Lyapunov** 算子，建立了与定理 **1.1** 相平行的结论。

定义 2.4（张维海, 浙江大学博士学位论文，1998; Liu Yazeng，山东大学博士论文，1999）. 线性时不变系统

$$\begin{cases} dx(t) = Ax(t)\,\mathrm{d}t + Cx(t)dW(t),\ x(0) = x_0 \\ y(t) = Dx(t) \end{cases}$$

是精确能观（exactly observable）的或简称 (A,C|D) 是精确能观的，意指

$$y(t) = 0,\ a.s.,\ t \in [0,T],\ \forall \mathrm{T} > 0 \Rightarrow x_0 = 0$$

定义 2.5.（W. Zhang，et al, IEEE TAC, 2008) (A,C|D) 是精确能检测（exactly detectable）的，意指

$$y(t) = 0,\ a.s.,\ t \in [0,T],\ \forall \mathrm{T} > 0 \Rightarrow \lim_{t\to\infty} E\|x(t)\|^2 = 0.$$

定理 2.2.（Popov-Belevith-Hautus 判据）(A,C|D) 是精确能观的充分必要条件是不存在非零的 $X \in S^n$，使下式成立：

$$\mathrm{L}_{A,C}X = AX + XA' + CXC' = \lambda X,\ DX = 0,\ \lambda \in \mathbb{C}$$

定理 2.3. （**Popov-Belevith-Hautus 判据**）**(A,C|D)** 是精确能检测的充分必要条件是不存在非零的 $X \in S^n$, 使下式成立：

$$\mathrm{L}_{A,C}X = AX + XA' + CXC' = \lambda X,\ DX = 0,\ \mathrm{Re}(\lambda) \geq 0.$$

除了精确能检测性的定义(定义2.4)，其他学者还引入了另外一种定义（通过能稳性对偶），称为随机能检测性（**stochastic detectability**）。

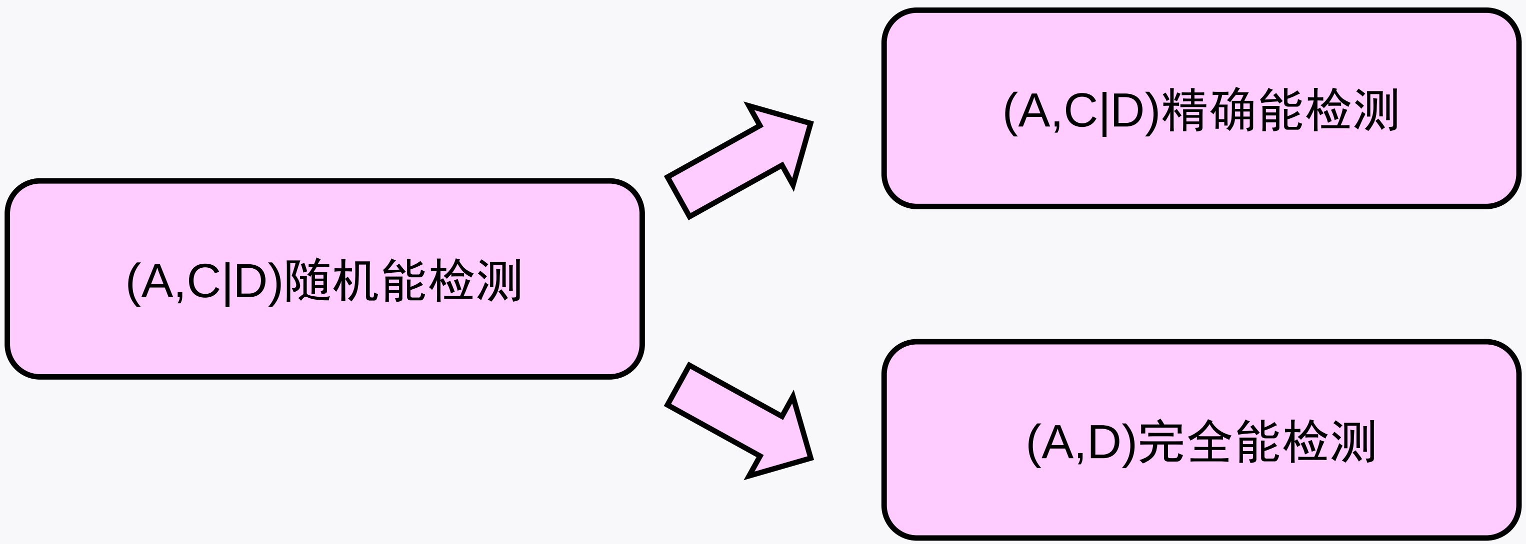
定义 2.5 （**T. Damm，2004，V. Dragan, et al,1997**）**. (A,C|D)** 称为是随机能检测的，如果存在一个实矩阵K, 使之

$$dx(t) = (A + KD)x(t)dt + Cx(t)dW(t),\ x(0) = x_0$$

是渐近均方稳定的。

利用定理 2.3， 我们可以证明： **(A,C|D)**的精确能检测性，随机能检测性以及**(A,D)**的完全能检测性（经典的能检测性），具有下列包含关系：

2.2 精确能观性和能检测性的谱刻画



但是 (A,C|D) 精确能检测与(A, D)完全能检测之间不存在包含关系

例 2.1. 取矩阵 A, C, D 如下：

$$\boldsymbol{A}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \boldsymbol{D}=\begin{bmatrix}0 & 1\end{bmatrix}, \mathrm{C}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

则利用定理 2.3-PBH 判据， 易验证（A,C|D）是精确能检测的。同样，容易验证不存在任何 $\boldsymbol{K}=\begin{bmatrix}\boldsymbol{k}_1 & \boldsymbol{k}_2\end{bmatrix}'$，使 （A+KD）为 Hurwitz 矩阵（稳定矩阵），因此 (A,D）非完全能检测的。

例 2.2. 取矩阵 A, C, D 如下：

$$\boldsymbol{A}=-\boldsymbol{I}_{2\times 2}, \boldsymbol{D}=\begin{bmatrix}0 & 1\end{bmatrix}, \boldsymbol{C}=2\boldsymbol{I}_{2\times 2}.$$

由于 A 是稳定的矩阵，因此对于任何的矩阵 D, (A,D) 是完全能检测的。 但是利用 PBH 判据，下列方程存在非零的解 $X=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \lambda=2$,因此（A,C|D）非精确能检测的。

$$\mathrm{L}_{A,C}\mathrm{X}=\mathrm{AX}+\mathrm{XA}'+\mathrm{CXC}'=\lambda\mathrm{X}, \boldsymbol{D}\mathrm{X}=0, \mathrm{Re}(\lambda)\geq 0.$$

定义 2.6. 二阶矩 Lyapunov 指数：

$$L_e^2 := \lim_{t\to\infty}\sup[\tfrac{1}{t}\log E\|x(t)\|^2]$$

定理 2.4.(**Zhang, Xie, IEEE TAC, 2009**) 若

$$\sigma(L_{A,C}) = \{\lambda : -\beta < \mathrm{Re}(\lambda) < -\alpha, \alpha, \beta > 0\}$$

则

（i）系统

$$dx(t) = Ax(t)dt + Cx(t)dW(t),\ x(0) = x_0$$

是指数均方稳定的，收敛速度比 $O(e^{-(\alpha+\varepsilon)t})$ 快，比 $O(e^{-(\beta+\varepsilon)t})$ 慢。

(ii) $-\beta < L_e^2 < -\alpha$ ， $L_e^2 = \max_{1\le i\le n(n+1)/2}\mathrm{Re}(\lambda_i)$.

随机系统

$$dx(t)=[Ax(t)+\mathrm{Bu(t)}]dt+[Cx(t)+\mathrm{Du(t)}]dW(t),\ x(0)=x_0$$

的极点配置该怎样定义？

定义 2.7. 若对复平面上的任意（**n+1)/2** 个点 $\lambda_1,\lambda_2,\cdots,\lambda_{n(n+1)/2}$，我们都可以找到一个反馈增益矩阵 **K**，使闭环算子的谱

$$L_{A,B;C,D}^{K}: X\in S^n \mapsto (A+BK)X+X(A+BK)'+(C+DK)X(C+DK)'\in \mathrm{S}^n$$

满足

$$\sigma(L_{A,B;C,D}^{K})=\{\lambda_1,\lambda_2,\cdots,\lambda_{n(n+1)/2}\}.$$

显然，$L^K_{A,B;C,D}$ 的伴随算子是

$$L^{K*}_{A,B;C,D}: X\in S^n \mapsto X(A+BK)+(A+BK)'X+(C+DK)'X(C+DK)\in S^n$$

定义 2.8. 我们称λ 是随机控制系统的一个不可移动的谱，如果存在非零的$X\in S^n$，使对任何的反馈增益矩阵 K，都有

$$L^K_{A,B;C,D}(X)=\lambda X, \text{或者 } L^{K*}_{A,B;C,D}(X)=\lambda X.$$

定理 2.5. λ 是一个不可移动的谱的充分必要条件是存在非零的$X\in S^n$，下列方程式同时成立：

$$\begin{aligned} &XA+A'X+C'XC=\lambda X,\\ &XB+C'XD=0,\\ &D'XD=0 \end{aligned} \qquad (2.1)$$

评注：若 $\sigma(L^{K}_{A,B;C,D})\subset\mathbb{C}^{-}$，随机系统称为能稳的，或简称(A,B;C,D)是能稳的。

我们知道：确定性线性系统 （A,B）是能稳的充分必要条件是 $L^{K}_{A,B}$ 不可移动的谱都在左半复平面内。下面的例子证明随机系统这一结论不成立。

例 2.3. 考虑一维系统。 取 $D\neq 0$，则不存在非零的$X\neq 0$ 满足（2.1），即在$D\neq 0$的情形下，系统不存在不可移动的谱。而系统是能稳的充分必要条件是

$$B^{2}+2BCD-2AD^{2}>0, \tag{2.2}$$

显然，（2.2）式不等价于$D\neq 0$。

确定性系统的完全能控性等价于$(A+BK)$的特征值能够任意配置，但是随机系统在精确终端能控的条件下，$L^{K}_{A,B;C,D}$的谱也不能任意配置。 例 2.3 是精确终端能控的， 但是谱不能任意配置，因为

$$\sigma(L^{K}_{A,B;C,D})=\lambda\geq 2A-\frac{(CD+B)^{2}}{D^{2}}+C^{2}$$

➢ S.Peng. Backward stochastic differential equation and exact controllability of stochastic control systems, Progress in Natural Science, 4, 274-284, 1994.

随机LQ 导致的广义代数 Riccati 方程：

$$\begin{aligned} & PA + A'P + C'PC + Q - (PB + C'PD) \\ & \qquad (R + D'PD)^{-1}(B'P + D'PC) = 0, \\ & R + D'PD > 0 \end{aligned} \tag{2.3}$$

定义 2.9. $P \in S^n$ 称为是（2.3）的反馈镇定解如果 $\sigma(L^K_{A,B;C,D}) \subset \mathbb{C}^-$；称为是强解（strong solution）如果 $\sigma(L^K_{A,B;C,D}) \subset \mathbb{C}^{-,0}$。

评注：de Souza (Systems and Control Letters, 14, 233-239, 1990) 用 $\sigma(A+BK) \subset \mathbb{C}^-$ 来定义（2.3）的强解，这是错误的，仅用 漂移项矩阵无法刻画随机稳定性。

定理 2.6（比较定理）．设 （A,B; C,D）能稳的，$(\boldsymbol{Q},\boldsymbol{R})\in \boldsymbol{S}^n\times\boldsymbol{S}^n$ ． 设 $\hat{\boldsymbol{P}}$ 是广义代数 Riccati 方程的实对称解。

$$\boldsymbol{PA}+\boldsymbol{A}'\boldsymbol{P}+\boldsymbol{C}'\boldsymbol{PC}+\widehat{\boldsymbol{Q}}-(\boldsymbol{PB}+\boldsymbol{C}'\boldsymbol{PD})(\widehat{\boldsymbol{R}}+\boldsymbol{D}'\boldsymbol{PD})^{-1}(\boldsymbol{B}'\boldsymbol{P}+\boldsymbol{D}'\boldsymbol{PC})=0, \quad (2.4)$$

$$\widehat{\boldsymbol{R}}+\boldsymbol{D}'\boldsymbol{PD}>0$$

如果 $\boldsymbol{R}\geq\widehat{\boldsymbol{R}},\boldsymbol{Q}\geq\widehat{\boldsymbol{Q}}$，则广义代数 Riccati 方程 （2.3）有一个最大解 $\bar{\mathrm{P}}, \bar{\boldsymbol{P}}\geq\widehat{\boldsymbol{P}}$.
而且， $\bar{\mathrm{P}}$ 是一个强解。

猜想： 强解也是最大解。

➢W. Zhang, H. Zhang, B.S. Chen. Generalized Lyapunov Equation Approach to State-Dependent Stochastic Stabilization/Detectability Criterion, IEEE TAC, 53(7), 1630-1642, 2008.

第三部分

H-表示技术及其应用

3.1. 背景（background）

熟知对 SDE

$$dx(t) = Ax(t)dt + Cx(t)dW(t),\ x(0) = x_0$$

令

$$X(t) = E[x(t)\,x(t)']$$

用 Ito 公式，

$$\begin{cases} \dot{X}(t) = L_{A,C}(X) = AX(t) + X(t)A^T + CX(t)C^T, \\ X(0) = x_0 x_0^T. \end{cases}$$

对上述对称矩阵方程，利用 Kronecker 积理论，可以得到下列的向量方程：

$$\dot{\vec{X}}(t) = [A \otimes I + I \otimes A + C \otimes C]\vec{X}(t),\ \vec{X}(0) = \overrightarrow{x_0 x_0^T}. \qquad (3.1)$$

问题 3.1. 方程 （3.1）是个标准的线性系统吗？

问题 3.2. 如何将 （3.1）转化为一个标准的线性系统？

3.1. 背景（background）

问题 3.1 的回答： No. （3.1）不是一个标准的线性系统。那么， 标准线性系统的定义是什么？

我们引用下列教材的 Definition 3.1 如下：

- **W. L. Brogan: Modern Control Theory, 3rd, 1990.**

“The state variables of a system consist of a minimum set of parameters which completely summarize the system’s status”.

3.1. 背景（background）

例 3.1. 令

$$A=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}, C=\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix}, x(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix},$$

则

$$X(t)=\begin{bmatrix}Ex_1^2(t) & E[x_1(t)x_2(t)]\\ E[x_2(t)x_1(t)] & Ex_2^2(t)\end{bmatrix}=\begin{bmatrix}x_{11} & x_{12}\\ x_{12} & x_{22}\end{bmatrix},$$

$$A\otimes I=\begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 1 & 0 & 2\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}, I\otimes A=\begin{bmatrix}1 & 2 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 1 & 1\end{bmatrix}, C\otimes C=\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 2 & 0 & 0\\ 2 & 0 & 2 & 0\\ 4 & 4 & 4 & 4\end{bmatrix},$$

此时方程（3.1）变为：

$$\dot{\vec{X}} = \begin{bmatrix} 3 & 2 & 2 & 0 \\ 3 & 4 & 0 & 2 \\ 3 & 0 & 4 & 2 \\ 4 & 5 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{12}(t) \\ x_{22}(t) \end{bmatrix}$$

表面看是个4-阶方程，实际上是个 3-阶方程，因为第二和第三个状态变量相同！

例 3.2.

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{12} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{12} \\ x_{22} \end{bmatrix} = Ax.$$

$$0 \in \sigma(A), \text{but} \lim_{t\to\infty} x(t) = 0.$$

结论： **Hurwitz** 稳定性判据仅仅对标准线性系统成立！

历史上， D.L.Kleinman (IEEE TAC, 14(4),1969) 曾经想当然地认为(3.1)的系数矩阵

$$\Gamma := [\boldsymbol{A} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{A} + \boldsymbol{C} \otimes \boldsymbol{C}]$$

一定有重的特征值，下面的反例证明这一断言是错误的。

例 3.3.

$$\boldsymbol{A} = \begin{bmatrix} -3 & 1/2 \\ -1 & -1 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

计算得知（Zhang and Xie, IEEE TAC, 2009）

$$\sigma(\Gamma) = \{-3+\boldsymbol{i}, -3-\boldsymbol{i}, -2, -4\}$$

没有重的特征值！

另外的问题，由第二节的 定理 2.1，

$$(\boldsymbol{A},\boldsymbol{C})\text{稳定} \;\Leftrightarrow\; \sigma(\mathrm{L}_{\boldsymbol{A},\boldsymbol{C}})\subset\mathbb{C}^{-},$$

但是，

$$\sigma(\boldsymbol{L}_{\boldsymbol{A},\boldsymbol{C}})=\{\lambda_1,\lambda_2,\cdots,\lambda_{\boldsymbol{n}(\boldsymbol{n}+1)/2}\},$$

$$\sigma[\boldsymbol{A}\otimes\boldsymbol{I}+\boldsymbol{I}\otimes\boldsymbol{A}+\boldsymbol{C}\otimes\boldsymbol{C}]=\{\lambda_1,\lambda_2,\cdots,\lambda_{\boldsymbol{n}(\boldsymbol{n}+1)/2},\cdots,\lambda_{\boldsymbol{n}^2}\}$$

并且，

$$\sigma(\boldsymbol{L}_{\boldsymbol{A},\boldsymbol{C}})\subset\sigma(\mathrm{A}\otimes\mathrm{I}+\mathrm{I}\otimes\mathrm{A}'+\mathrm{C}\otimes\mathrm{C})$$

问题：

$$(\boldsymbol{A},\boldsymbol{C})\text{稳定}\Rightarrow\sigma(\mathrm{A}\otimes\mathrm{I}+\mathrm{I}\otimes\mathrm{A}+\mathrm{C}\otimes\mathrm{C})\subset\mathbb{C}^{-}?$$

$\boldsymbol{C}=0$ 情形是正确的，$\boldsymbol{C}\neq0$，至今尚未举出反例。文献上当成理所当然的。

为了回答问题 3.2，即如何将非标准的线性系统

$$\dot{\vec{X}}(t)=[A\otimes I+I\otimes A+C\otimes C]\vec{X}(t)$$

转化为标准系统，引进 H-表示技术：

设 $\mathbb{X}$ 是一个 p-维的方阵空间， $e_1,e_2,\cdots,e_p$ 是 $\mathbb{X}$ 的一组基，则

$$X=\sum_{i=1}^{p}x_i e_i.$$

如果 X 是实矩阵，则 $x_i\in\mathcal{R}$ ；如果 X 为复矩阵，则 $x_i\in\mathbb{C}$ 。对上述方程拉直运算（张量），有

$$\vec{X}=\begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_p\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_p\end{bmatrix}=H\tilde{X}.$$

定义 3.1. $H\tilde{X}$ 称为是 $\vec{X}$ 的 H-表示。

例 3.4. $\mathbb{X}=\boldsymbol{S}^2$，二阶对称矩阵空间。设 $X=(x_{ij})_{2\times 2}\in\mathbb{X}$，则 $\dim(\mathbb{X})=3$，选标准基如下：

$$\boldsymbol{e}_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},\boldsymbol{e}_2=\begin{bmatrix}0&1\\1&0\end{bmatrix},\boldsymbol{e}_3=\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

则 $\vec{\boldsymbol{X}}=\boldsymbol{H}\tilde{\boldsymbol{X}}$ 的表示如下：

$$\vec{\boldsymbol{X}}=\begin{bmatrix}\boldsymbol{x}_{11}\\\boldsymbol{x}_{12}\\\boldsymbol{x}_{12}\\\boldsymbol{x}_{22}\end{bmatrix},\tilde{\boldsymbol{X}}=\begin{bmatrix}\boldsymbol{x}_{11}\\\boldsymbol{x}_{12}\\\boldsymbol{x}_{22}\end{bmatrix},\boldsymbol{H}=\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\\0&0&1\end{bmatrix}.$$

例 3.5. $\mathbb{X}=\boldsymbol{S}^{-2}$ ，二阶反对称矩阵空间。设 $X=(x_{ij})_{2\times2}\in\mathbb{X}$，则 $\dim(\mathbb{X})=1$，选标准基如下：

$$\boldsymbol{e}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$

则 $\vec{\boldsymbol{X}}=\boldsymbol{H}\tilde{\boldsymbol{X}}$ 的表示如下：

$$\vec{\boldsymbol{X}}=\begin{bmatrix}0\\ \boldsymbol{x}_{12}\\ -\boldsymbol{x}_{12}\\ 0\end{bmatrix},\ \tilde{\boldsymbol{X}}=\boldsymbol{x}_{12},\ \boldsymbol{H}=\begin{bmatrix}0\\ 1\\ -1\\ 0\end{bmatrix}.$$

可以证明：对任何空间 $\mathbb{X}$ ，$\boldsymbol{H}$ 矩阵都是列满秩的。

现在利用 H-表示可以将非标准线性系统：

$$\dot{\vec{X}}(t)=[A\otimes I+I\otimes A+C\otimes C]\vec{X}(t)$$

转化为标准线性系统：

$$H\dot{\tilde{X}}(t)=[A\otimes I+I\otimes A+C\otimes C]H\tilde{X}(t)$$

上式左乘以 H' ，

$$H'H\dot{\tilde{X}}=H'(A\otimes I+I\otimes A+C\otimes C)H\tilde{X},$$

由于 H 列满秩， $(H'H)^{-1}$ 存在， 所以

$$\begin{cases}\dot{\tilde{X}}=(H^TH)^{-1}H^T(A\otimes I+I\otimes A+C\otimes C)H\tilde{X}\\ \quad=H^+(A\otimes I+I\otimes A+C\otimes C)H\tilde{X}=\tilde{H}\tilde{X},\\ \tilde{X}(0).\end{cases}\qquad(3.2)$$

这样我们就将非标准的线性系统(3.1)转化为标准的线性系统(3.2)。在(3.2)中，

$\boldsymbol{H}^{+}=[\boldsymbol{H}^{T}\boldsymbol{H}]^{-1}\boldsymbol{H}^{T}$ 是 $\boldsymbol{H}$ 的 Moore-Penrose 逆，

$$\tilde{\boldsymbol{H}}=\boldsymbol{H}^{+}(\boldsymbol{A}\otimes\boldsymbol{I}+\boldsymbol{I}\otimes\boldsymbol{A}+\boldsymbol{C}\otimes\boldsymbol{C})\boldsymbol{H}.$$

定理 3.1. (i) (A,C)是稳定的充分必要条件是 $\sigma(\tilde{\boldsymbol{H}})\subset\mathcal{C}^{-}$，这里 $\tilde{\boldsymbol{H}}$ 是一个 $\frac{\boldsymbol{n}(\boldsymbol{n}+1)}{2}\times\frac{\boldsymbol{n}(\boldsymbol{n}+1)}{2}$ -的矩阵。(ii) $\sigma(\tilde{\boldsymbol{H}})=\sigma(\mathrm{L}_{A,C})$.（算子的谱和矩阵的特征值相同）

评注： 利用 H-表示技术可以将很多随机控制的问题转化为确定性系统求解。特别是在二阶矩稳定性、时变系统的能观性、能检测性等。

3.3. 应用之一：非线性矩稳定

- H-表示技术可以用于非线性随机时滞系统的矩稳定性研究，见下面两篇文章：

- X. Zhao. F. Deng. Moment stability of discreat stochastic systems with time-delays based on H-representation technique, Automatica, 50(2), 530-536, 2014.

- X. Zhao. F. Deng. Moment stability of nonlinear stochastic systems with time-delays based on H-representation technique, IEEE TAC, 59(3), 814-819, 2014

- H-表示技术可以用于研究广义 Lyapunov 方程：

$$PA + A^T P + C^T PC = -Q. \tag{3.3}$$

特殊解结构的存在性。

除了对称算子 $L_{A,C}$ 外，我们再引进一个反对称算子 $L^-_{A,C}$ 如下：

$$L^-_{A,C} : X \in S^{-n} \mapsto AX + XA' + CXC'$$

$$\tilde{H}_- := H_-^+(A \otimes I + I \otimes A + C \otimes C)H_-$$

H_- 是 $X \in S^{-n}$ 的 H-表示矩阵。

定理 3.2. 对于任何对称矩阵 $Q \in S^n$，广义 Lyapunov 方程(3.3)有唯一对称解的充分必要条件是：$0 \notin \sigma(L_{A,C})$.

3.3. 应用之二：广义 Lyapunov 方程

三、H-表示技术及其应用

定理 3.3. 对于任何反对称矩阵 $Q \in S^{-n}$，广义 Lyapunov 方程有唯一反称解的充分必要条件是：$0 \notin \sigma(L^{-}_{A,C})$.

评注： $0 \notin \sigma(L_{A,C}) / 0 \notin \sigma(L^{-}_{A,C})$ 等价于 $\det(\tilde{H}) \neq 0 / \det(\tilde{H}_{-}) \neq 0$，用来保证$Q \in S_n (Q \in S_{-n})$ 情形下广义 Lyapunov 方程 (3.3）有唯一的对称（反对称）解。但是此时并不排除其他解的存在性。

评注： W. L. Brogan （ Modern Control Theory, 3rd , 1991 ）断言：如果 ，则$Q \in S_n$广义 Lyapunov 方程（3.3）若有解则一定是对称解。下面的反例证明这一断言是错误的。

例3.6. 我们只要看经典的 Lyapunov 方程，即 $\boldsymbol{C}=0$，$\boldsymbol{Q}=0$ ，而

$$\boldsymbol{A}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

可以验证有无穷多个解

$$\boldsymbol{P}=\begin{bmatrix}0 & \boldsymbol{x}_{12}\\ \boldsymbol{x}_{21} & 0\end{bmatrix},$$

当 $\boldsymbol{x}_{12}=\boldsymbol{x}_{21},\boldsymbol{P}\in\boldsymbol{S}^{2}$；当 $\boldsymbol{x}_{12}=-\boldsymbol{x}_{21}$ ，$\boldsymbol{P}\in\boldsymbol{S}^{-2}$ ；其他情况下，$\boldsymbol{P}$ 既不属于对称矩阵也不属于反对称矩阵。

下列时变线性系统的能观性和定常系统（时不变）不一样。

$$\begin{cases} dx(t) = A(t)x(t)dt + C(t)x(t)dw(t) \\ x(0) = x_0 \in R^n \\ y(t) = D(t)x(t) \end{cases} \quad (3.4)$$

定义 3.2. 系统(3.4)在时刻 $t_0 \in R^+$是能观的，如果存在一个有限的时间区间 $[t_0, t_1]$，$t_1 > t_0$，当 $y(t) \equiv 0, a.s., \Rightarrow x(t_0) = 0, a.s.$.

如果系统在每个时间 $t_0 \in R^+$都是能观的，则系统是完全能观的。

利用 H-表示，可以将随机系统（3.4）的能观性研究转化为确定性时变系统的研究：

$$\begin{cases} \dot{\tilde{X}}(t) = \tilde{H}(t)\tilde{X}(t), \ \tilde{H}(t) = H^+(A(t)\otimes I + I\otimes A(t) + D(t)\otimes D(t))H, \\ \tilde{Y}(t) = N(t)\tilde{X}(t), \ N(t) = H^+(\mathrm{D(t)}\otimes D(t))H. \end{cases} \quad (3.5)$$

三、H-表示技术及其应用

令 $\Phi(t,t_0)$ 是确定性时变系统(3.5)状态转移矩阵，即

$$\dot{\Phi}(t,t_0)=\tilde{H}(t)\Phi(t,t_0),\ \Phi(t_0,t_0)=I_{n\times n}.$$

定理 3.4. 随机时变系统在 t_0 是能观的充分必要条件是存在 $t_1>t_0$，使 Grammian 矩阵

$$W(t,t_0)=\int_{t_0}^{t_1}\Phi'(t,t_0)N'(t)N(t)\Phi(t,t_0)\,dt$$

是非奇异的， 即 $W(t,t_0)$ 为正定的。

推论： 当系统(3.4)为时不变系统时， 系统是完全能观的（精确能观的）充分必要条件是 $Rank(\Delta)=n(n+1)/2,$ 其中，

$$\Delta=\begin{bmatrix} N' & (N\tilde{H})' & \dots & (N\tilde{H}^{n(n+1)/2-1})' \end{bmatrix}'.$$

3.4. 总结

H-表示技术是一个有用的工具，它能够将某些随机控制问题转化为确定性系统的问题进行求解，从而可以充分利用已有的结论. 其次，不仅限于矩阵空间，其他有限维空间去掉冗余度，获取有用信息，降低维数也同样有效。最近，我们发现 H-表示和张量积之间的联系，并用来研究 p-阶矩的稳定性和 p-阶矩的能观性。

- H. Zhang, J. Xia, **W. Zhang**, et al. pth moment asymptotic stability/stabilization and pth moment observability of linear stochastic systems: generalized H-representation, IEEE Trans. TSMC, accepted for publication.

第四部分

国内外学者的典型引用和评价

四. 国内外公开评价

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An iterative algorithm to solve state-perturbed stochastic algebraic Ricca[ti] equations in LQ zero-sum games

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四. 国内外公开评价

direct methods cannot solve some H_∞ AREs well (see Example 3 in [3]). Since AREs considered in [2,3] are from linear deterministic systems, a natural question, and indeed the one arising here, is: "can we extend the algorithm to linear stochastic systems?" The immediate answer is yes. However, LQ stochastic game problems are more complicated than LQ deterministic game problems. For example, in LQ stochastic game problems, noises can enter state, control input and unknown disturbance input (see [7,1] and references therein). Also, Riccati equations with random coefficients arise in Kalman filter and LQG controller design (see [8,9] and references therein). Recently, stochastic differential games with state-dependent noises have attracted much attention and have been widely applied to various fields (for example to stochastic H_∞ in [10–12]). Motivated by these applications, we focus on developing a new algorithm to solve the SARE (1) arising in LQ stochastic differential games with state-dependent noises.

There exist many algorithms to solve the deterministic AREs. However, when the state-dependent noise is introduced into a LQ deterministic game problem, in order to obtain the saddle point solutions for each player, one needs to solve (1). This equation is typically more difficult to solve than AREs in LQ deterministic game problems because of the additional disturbance term reflected in (1). Newton's method can be used to solve (1) (see [11, 12] for example); however, a serious issue is that in Newton's method, one must choose a suitable initial point (i.e. convergence assuring) to implement an algorithm and such an initial point is not always straightforward to obtain.

In contrast, in our algorithm, we can always choose a particularly simple initial point, viz $P_0 = 0$, and this is a major advantage of our algorithm. In [13], a recursive algorithm is developed to solve H_2 SAREs (i.e. the quadratic term in SAREs is negative semidefinite); however, this algorithm cannot be used to solve SARE (1), which has a sign indefinite quadratic term. So our task is to develop a new recursive algorithm, which can be used to solve SARE (1) and this work is motivated by the algorithm in [2,3]. Our algorithm replaces the task of solving a SARE (1) by the task of solving a sequence of SAREs with negative semidefinite quadratic terms; then by using the algorithm in [13], we can solve these SAREs recursively. We prove that the solution of the original SARE can be obtained by using the solutions of these H_2-type SAREs. In some sense therefore, our work in this paper is an extension of the work in [2,3] since it provides an algorithm to solve SARE (1) which is more general than the AREs considered in [2,3]. Note also that the idea in [2,3] is closely related to the method in [14–16]. In [14–16], it is the proved that the existence of a stabilizing solution for an indefinite ARE is equivalent to the existence of stabilizing solutions of two definite AREs; however, as pointed out in [14], the method in [14–16] is only used to prove the existence of the stabilizing solution for an indefinite ARE, not for computation.

Another motivation of this paper comes from the work in [13,17]. In [13,17], a SARE with a negative semidefinite quadratic term is considered to solve an optimal control problem associated with a kind of linear stochastic systems where the system's parameters are deterministic but state-dependent noises are included. In [13,17], a sufficient condition is obtained for the existence of unique positive semidefinite and stabilizing solutions of such SAREs; also, an iterative algorithm to solve such SAREs is developed in [13,17]. Hence the question of how to extend the algorithm in [13,17] to solve SAREs with a sign indefinite quadratic term arises here. In some sense, the algorithm to solve SAREs with a negative semidefinite quadratic term in [13,17] can be regarded as an extension of the Kleinman algorithm in [6], since an additional disturbance term is reflected in such SAREs.

There are some conceptual challenges when state-dependent noises enter linear systems. For example, for our algorithm in the LTI case (i.e. the algorithm in [2,3]), stabilizability and detectability is required to implement our algorithm; so the question arises here of how to define stabilizability and detectability when state-dependent noises enter systems. In this paper, we will review the definitions of these important control concepts in the literature as appropriate for a stochastic framework.

The paper is organized as follows: Section 2 gives some definitions and preliminary results, Section 3 presents our main result, Section 4 states the algorithm, Section 5 presents a game theoretic interpretation, Section 6 provides a numerical example, Section 7 records our conclusions.

Notation. $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices; $\mathbb{Z}$ denotes the set of integers with $\mathbb{Z}_{\geq a}$ denoting the set of integers greater or equal to $a \in \mathbb{Z}$. Define function spaces as follows:

$$\mathcal{U} = \left\{u : \mathbb{R}^+ \to \mathbb{R}^m \middle| \int_{t_0}^{t_1} \|u(t)\|^2 dt < \infty \forall t_0, t_1 \in \mathbb{R}^+\right\},$$

$$\mathcal{Y} = \left\{y : \mathbb{R}^+ \to \mathbb{R}^r \middle| \int_{t_0}^{t_1} \|y(t)\|^2 dt < \infty \forall t_0, t_1 \in \mathbb{R}^+\right\}.$$

$\rho[\cdot]$ denotes the spectral radius of a square matrix; $\overline{\sigma}(\cdot)$ denotes the maximum singular value of a matrix; $\|\cdot\|$ denotes the Euclidean norm. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all its eigenvalues have negative real part. $E(\cdot)$ denotes the mathematical expectation.

2. Definitions and preliminary results

In this section, we will give some definitions and preliminary results.

To motivate the definitions in this section, we firstly define the following stochastic control system Δ

$\Delta : \mathcal{U} \to \mathcal{Y}$

这里引用了申请人在文献[18,19]中给出的定义。

value. The ma… $^{\times n}$, $\bar{A} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$ are all real. Wit… generality, throughout this paper, we assume w to be a …nsional standard Wiener process.

We first recall th…ept of stabilizability for such a system, which generalizes the …bilizability of deterministic systems to the stochastic context an… plays an important role in our algorithm.

Definition 1 (*[18,19]*). The system Δ is said to be stabilizable, briefly $(A, B|\bar{A})$ is stabilizable, if there exists a feedback control $u(t) = Kx(t)$, such that for any $x_0 \in \mathbb{R}^n$, the closed-loop system

$$dx(t) = (A + BK)x(t)dt + \bar{A}x(t)dw(t) \quad (6)$$

is asymptotically mean square stable, i.e.

$$\lim_{t\to\infty} Ex(t)x^T(t) = 0.$$

Here, K is a constant matrix with suitable dimensions. In addition, suppose $u(t)$ in (4) satisfies $u(t) \equiv 0$; if

$$dx(t) = Ax(t) + \bar{A}x(t)dw(t) \quad (7)$$

is asymptotically mean square stable; then $(A, \bar{A})$ is called stable.

We remark that for systems of the form $dx(t) = A_1x(t)dt + A_2x(t)dw(t)$, the property of asymptotic mean square stability

这里引用了申请人在文献[18,19]中给出的定义。

systems.

Definition 2 (*[18,1…* …, C, $\bar{A}$ be the matrices appearing in the system Δ; if there exi… …al matrix H with suitable dimensions, such that the closed-loop …stem

$$dx = (A + HC)xdt + \bar{A}xdw \quad (8)$$

is asymptotically mean square stable, then $(A, C|\bar{A})$ is called stochastically detectable.

Definition 3. Let A, $\bar{A}$, B_1, B_2, C be the m… …ing in Eq.

引用了申请人在文献[18]中给出的结论。

…ng

…ese

…d 9

below.

Definition 4. Let A, B_1, B_2, C be the real matrix fu… appearing in (1). Suppose there exists a positive se… stabilizing solution Π to (1). Let $P \in \mathbb{R}^{n\times n}$ be a sym… …x. Let $\hat{A}_p \in \mathbb{R}^{n\times n}$ be defined as

$$\hat{A}_p = A + B_1B_1^TP - B_2B_2^TP.$$

引用了申请人在文献[18,19]中给出的结论。

Lemma 5. *Let A, $\bar{A}$, B_1, B_2, C be the real matrix functions appearing in (1) and suppose P, $Z \in \mathbb{R}^{n\times n}$. Define*

$$F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$$
$$P \longmapsto PA + A^TP + \bar{A}^TP\bar{A} - P(B_2B_2^T - B_1B_1^T)P + C^TC. \quad (9)$$

If $P = P^T$ and $Z = Z^T$, then

$$F(P + Z) = F(P) + Z\hat{A}_p + \hat{A}_p^TZ + \bar{A}^TZ\bar{A} - Z(B_2B_2^T - B_1B_1^T)Z \quad (10)$$

where $\hat{A}_p$ is the matrix in Definition 4. If in addition, there holds

$$0 = Z\hat{A}_p + \hat{A}_p^TZ + \bar{A}^TZ\bar{A} - ZB_2B_2^TZ + F(P) \quad (11)$$

then

$$F(P + Z) = ZB_1B_1^TZ \quad (12)$$

and

$$\rho[F(P + Z)] = \overline{\sigma}(B_1^TZ)^2. \quad (13)$$

Proof. The first result can be obtained by direct computations; the second claim is then trivial. □

The next three lemmas (Lemmas 6–8) are known general results. The first of these gives sufficient conditions for the existence of a unique positive semidefinite and stabilizing solution for a class of SAREs.

Lemma 6 (*[18]*). *Consider the system Δ and assume that $(A, B|\bar{A})$ is stabilizable and $(A, C|\bar{A})$ is stochastically detectable. Then, there exists a positive semidefinite and stabilizing solution Z satisfying the following SARE*

$$0 = A^TZ + ZA + \bar{A}^TZ\bar{A} - ZBB^TZ + C^TC. \quad (14)$$

Furthermore, Z is the unique stabilizing solution of (14) (i.e. there is no other stabilizing solution to (14)).

Proof. See [18]. □

The n…t lemma recalls a standard result on the stability of linear …ochastic systems.

Lemma 7 (*[18,17,19]*). *Let A, $\bar{A}$, C be the real matrix functions appearing in the system Δ, and suppose that the pair $(A, C|\bar{A})$ is stochastically detectable. Then $(A, \bar{A})$ is stable if and only if the following Lyapunov-type equation:*

$$0 = PA + A^TP + \bar{A}^TP\bar{A} + C^TC \quad (15)$$

has a un…e positive semidefinite solution P.

P… See [18,17,19]. □

…e next lemma gives a uniqueness result regarding the stabilizing solution of (1).

Lemma 8. *Suppose there exists a stabilizing solution Π to (1); then this solution must be the unique stabilizing solution to (1) (i.e. there is no other stabilizing solution to (1)). Furthermore, if $\Pi \geq 0$, then the system $dx(t) = (A - B_2B_2^T\Pi)x(t) + \bar{A}x(t)dw_1(t)$ is asymptotically mean square stable.*

Proof. See [24]. □

The next lem…a sets up some basic relationships between the stabilizing sol…on Π to Eq. (1) when it exists and the matrix functions P, …atisfying Eq. (11). It is not standard, but is reminiscent o… …ilar results in [2,3,21–23]. It plays a crucial role in establishin… …alidity of the algorithm of the next section.

Lemma 9. *L… …B_1, B_2, C be the matrices appearing in (1), let $P = P^T \in$ … $Z = Z^T \in \mathbb{R}^{n\times n}$ satisfy Eq. (11), and let a stabilizing Π… $\mathbb{R}^{n\times n}$ satisfy Eq. (1). Let $\hat{A}_p$ be the matrix in Definition 4.*

(i) $\Pi \geq$ (… …) *is asymptotically mean square stable.*
(ii) $(\hat{A}_{P+Z}$, … …*lly mean square stable if $\Pi \geq (P + Z)$.*

这个引理的证明采用了申请人在文献[24]中给出的证明方法。

via (9), substituting (17) into (16) and rearranging, we have

$$0 = \Sigma\hat{A}_P + \hat{A}_P^T\Sigma + \Sigma B_2B_2^T\Sigma + (\Pi - P)B_1B_1^T(\Pi - P) + \bar{A}^T\Sigma\bar{A}, \quad (18)$$

where $\Sigma = \Pi - (P + Z)$. Then since $(\hat{A}_P, \bar{A})$ is asymptotically mean square stable, we conclude $\Pi \geq (P + Z)$ by using Lemma 7.

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四. 国内外公开评价

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Moment Stability of Nonlinear Stochastic Systems With Time-Delays Based on $\mathcal{H}$-Representation Technique

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In fact, [29] introduced some definitions and lemmas of $\mathcal{H}$-representation matrices. In addition, in order to conveniently prove our main theorems, we present some more definitions and lemmas of the $\mathcal{H}$-representation matrices in our own way.

Lemma 1 [11], [29]: There exists a rectangular $n^2 \times (\frac{n(n+1)}{2})$ transformation matrix $\overrightarrow{H}_n$ such that $x^{[2]} = \overrightarrow{H}_n x_{[2]}$ for $x \in R^n$.

Lemma 2[11], [29]: $\overrightarrow{H}_n^T \overrightarrow{H}_n$ is invertible.

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Brief paper

Moment stability of nonlinear discrete stochastic systems with time-delays based on $\mathcal{H}$-representation technique☆

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2.1. $\mathcal{H}$-representation matrices: related definitions and lemmas

The $\mathcal{H}$-representation technique is not well-known to the readers, so we firstly give a brief introduction to the $\mathcal{H}$-representation technique. The $\mathcal{H}$-representation technique is one of the basic tools for our stability analysis of discrete stochastic systems, with which we can reduce the dimension of the moment equation and the dimension of the related stability condition associated with the moment equation, then we can more conveniently determine stability properties of systems. In fact, an $\mathcal{H}$-representation matrix is just a matrix which can realize expanded or compressed transformation of a moment or a vector.

In fact, Zhang and Chen (2012) introduced some definitions and lemmas of H-representation matrices. In addition, in order to

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SEVERAL ITERATIVE PROCEDURES TO COMPUTE THE STABILIZING SOLUTION OF A DISCRETE-TIME RICCATI EQUATION WITH PERIODIC COEFFICIENTS ARISING IN CONNECTION WITH A STOCHASTIC LINEAR QUADRATIC CONTROL PROBLEM*

Vasile Drăgan†

罗马尼亚科学院高级研究员。

Dedicated to the memory of Prof. Dr. Viorel Arnăutu

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In the last part of the paper, we propose a method to compute the periodic solution occurring at each step of a Kleinman type algorithm. Our method is based on the so called H-representation technique recently developed in [21]. This method allows us to reduce the computation of the periodic solution of a Lyapunov type equation to the computation of the periodic solution of a backward affine eq ation on an euclidian space of dimension $n(n+1)/2$, n being the dimensio of the state space of controlled system under consideration. In the last sec of the paper, a comparison between several types of numerical methods d sed in the paper is done.

[21] W. Zhang, B. Chen. H-Representation and Applications to Generalized Lyapunov Equations and Linear Stochastic Systems. *IEEE Trans. on Aut. Contr.*57 (12): 3009–3022, 2012.

我们的方法是基于H表示技术。这种技术允许我们简化Lyapunov型方程周期解的计算。

4.2 The H-representation technique revisited

In this paragraph we briefly recall the method of H-representation of a Lyapunov operator in terms of a matrix on the space of dimension $\hat{n} = \frac{n(n+1)}{2}$. This allows us to rewrite the equation (28) in the form of an equation of type (31).

For details we refer to [21], where this method was introduced. We recall that if $X \in \mathbb{R}^{n\times n}$, then $\Psi(X) = Vec(X) = (x(1), x(2), ..., x(n))^T \in \mathbb{R}^{n^2}$ where $x(i)$ is the i^th line of the matrix X, $1 \le i \le n$.

Let $E_{11}, E_{12}, ..., E_{1n}, E_{22}, ..., E_{2n}, ..., E_{n-1n-1}, E_{n-1n}, E_{nn}$ be the standard base of the space of symmetric matrices $\mathcal{S}_n$.

This means that $E_{pq} = (e_{pq}(i,j))_{\overline{i,j=1,n}}$ with $e_{pq}(ij) = 1$ if $(ij) \in \{(pq),(qp)\}$ and $e_{pq}(ij) = 0$ otherwise. If $X \in \mathcal{S}_n$ is an arbitrary symmetric matrix, then

$$X = E_{11}x_1 + ... + E_{1n}x_n + E_{22}x_{n+1} + ... + E_{nn}x_{\hat{n}}. \tag{35}$$

We introduce the linear operator $\varphi : \mathcal{S}_n \to \mathbb{R}^{\hat{n}}$ defined by

$$\varphi(X) = x \tag{36}$$

where $x = (x_1, x_2, ..., x_{\hat{n}})^T$ is the vector whose components occur in the right hand side of (35). We introduce also the matrix

$$H = \begin{pmatrix} \Psi(E_{11}) & \Psi(E_{12}) & \ldots & \Psi(E_{1n}) & \Psi(E_{22}) & \ldots & \Psi(E_{n-1n}) & \Psi(E_{nn}) \end{pmatrix}.$$

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EXPONENTIAL STABILITY IN MEAN SQUARE OF A LARGE CLASS OF SINGULARLY PERTURBED STOCHASTIC LINEAR DIFFERENTIAL EQUATIONS *

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Toader Morozan[¶]

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Using the techniques of H-representation, introduced in [23] we obtain that $Z(\tau) = \varphi^{-1}(\xi(\tau))$ where $\varphi : \mathcal{S}_{n_2} \to \mathbb{R}^{\frac{n_2(n_2+1)}{2}}$ is the isomorphism introduced in [23] and $\xi(\tau)$ is the solution of the following problem with given initial values:

$$\frac{d}{d\tau}\xi(\tau) = \mathbf{\Theta}(t,i)\xi(\theta), \quad \xi(0) = \varphi(I_{n_2}), \tag{32}$$

where $\mathbf{\Theta}(t,i)$ is the matrix associated to the linear operator $\mathcal{L}_{f\nu_0}(t,i)$ via (9) from [23]. Invoking the boundedness of the functions $t \to A_{22}(t,i)$ and $t \to A_{k,22}(t,i)$ we may deduce that there exists $\check{\gamma} > 0$ not depending upon $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$ such that $|\mathbf{\Theta}(t,i)| \leq \check{\gamma}$.

On the other hand, from Lemma 3.1 (ii) from [23] applied in the case of linear operator defined in (29), we infer that the spectrum of the operator $\mathcal{L}_{f\nu_0}(t,i)$ coincides to the spectrum of the matrix $\mathbf{\Theta}(t,i)$.

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Brief paper

On detectability of stochastic systems☆

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Stochastic linear control systems have attracted consi interest in the last 40 years. In the framework of Itô eq together with the notion of mean-square stability it ha observed that problems of e.g. linear quadratic and H^∞ can be treated quite analogously to the deterministic case, leading to generalized, but similar Riccati-type matrix equations or corresponding LMI-problems (see for instance Hinrichsen Pritchard, 1998; Petersen, Ugrinovskii, & Savkin, 2000; Wonham, 1970; Yong & Zhou, 1999). Still there seems to be an ongoing struggle for the appropriate concepts of and relations between stabilizability and detectability, e.g. in Drăgan, Halanay, and Stoica (1997), Tessitore (1997), Freiling and Hochhaus (2004), Zhang and Chen (2004).

（学者）似乎仍然在继续探索能稳性、能检测性的适当定义以及者之间的关系，例如，见文献 …., 张和陈（2004）。

Usually one defines a system to be detecta if all its unstable modes produce a non-zero output, i.e. if van the output $y(t)=0$ for all t implies that the state $x(t)$ con to zero.

In the deterministic case, it follows that a syste if and only if the dual system is stabilizable, w

W. Zhang, B. S. Chen. On stabilizability and exact observability of stochastic systems with their applications, Automatica, 40(1), pp.87-94, 2004.

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor René Boel

ome non-zero output is only a necessary but not ient condition for stabilizability of the dual system, and the latter does not give rise to a practicable observer equation. Moreover, there is a purely algebraic way to define an analogue of the Hautus-test for stochastic systems, which turns out to be a necessary condition for stabilizability of the dual system.

Each of these properties may thus be taken as a starting point to define detectability in the stochastic case. Several authors (e.g. Da Prato & Ichikawa, 1985; Drăgan et al., 1997; Fragoso, Costa, & de Souza, 1998; Freiling & Hochhaus, 2004; Tessitore, 1997) have chosen the second, i.e. stabilizability of the dual tem, as a defining property. This choice, however, has some Firstly, there is no clear interpretation with respect rol is or ed algebraic Lyapunov and Riccati equations only the generalized Hautus-test is used, which is weaker than stabilizability of the

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张和陈在 2004 年的文章中采用了一个类似的方法，他们称一个系统是精确能观的，如果系统的所有非平凡解（不仅是非稳定解）能产生非零的输出。他们得到了一个等价的 Hautus-检验，这是一个确定性系统 Hautus-检验的类比。……，作为应用，我们加强了张和陈 2004 年关于广义 Lyapunov 方程的一个结果。

$$\sum_{j=1}^{N} C_j^{\mathrm{T}} u \,\mathrm{d}w_j$$

Moreover, an … given that detectability in this sense does … always imply stabilizability of the dual system.

A similar approach has been taken by Zhang and Chen (2004), who call a system exactly observable if all non-trivial solutions (not only the unstable ones) cause some non-zero output; they derive an equivalent Hautus-test, which again is an analogue of the Hautus-test for observability of deterministic systems. Our notion of detectability fits nicely into this framework, since it is weaker than (exact) observability, as one would expect. As an application we strengthen a result on generalized algebraic Lyapunov equations given in Zhang and Chen (2004).

2. Definition of detectability

We consider stochastic linear systems of the form

is stabilizable. This means (e.g. Tessitore, 1994; Willems & Willems, 1976) that there exists a gain matrix K such that the closed-loop system

$$\mathrm{d}x = (A + KC)^{\mathrm{T}} x \,\mathrm{d}t + \sum_{j=1}^{N} (A_j + KC_j)^{\mathrm{T}} x \,\mathrm{d}w_j \qquad (2)$$

is mean-square stable. An observer, based on this equation would take the form of system (1) with additional input

$$\mathrm{d}\xi = A\xi \,\mathrm{d}t + \sum_{j=1}^{N} A_j \xi \,\mathrm{d}w_j + K(\mathrm{d}\eta - \mathrm{d}y),$$

$$\mathrm{d}\eta = C\xi \,\mathrm{d}t + \sum_{j=1}^{N} C_j \xi \,\mathrm{d}w_j.$$

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为了比较，我们回顾一下精确能观性的定义。该定义由张和陈 2004 年对输出方程不含噪音项的情形所给出，但是可以直接推广到更一般的情况。

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Since $\lambda \geqslant 0$, at least one of the summands must not converge to zero for $t \to \infty$, i.e. for some ℓ_0 we have

$E\|x($ …

On t… and …

$C_j x(t, x_0$ … Hence $y(t, x_0^{(\ell_0)})$ is constant and can be assumed to be zero (since we can assume w.l.o.g that $y(0, x_0^{(\ell_0)}) = 0$). Therefore the system is not detectable.

Let us now assume that the criterion of the theorem holds and there exists a non-zero solution $x(t, x_0)$ with $E\|y(t, x_0)\|^2 = 0$ for all $t \geqslant 0$. It follows that $C_j x(t, x_0) = 0$ for all $t \geqslant 0$ and $j = 0, \ldots, N$. We need to show that

$$E\|x(t, x_0)\|^2 \to 0 \quad \text{as } t \to \infty. \tag{6}$$

The second-moment matrix $P(t) = Ex(t, x_0)x(t, x_0)^{\mathrm{T}}$ satisfies

System (7) is mean-square stable, if and only if

$$\beta(L_{A+KC} + \Pi_K) < 0. \tag{8}$$

… nd $CX = 0$. Then

… λX,

…

For a comparison we recall the definition of exact observability. It was defined in Zhang and Chen (2004) for the case without noise terms in the output equation, but it is straightforward to restate the condition in our more general situation.

Definition 10. System (1) is *exactly observable*, if $y(t) \neq 0$ for all $t \geqslant 0$ almost surely, implies $x_0 \neq 0$.

The corresponding version of the Hautus test is the following.

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Nevertheless, the given system is detectable. In fact, $CX \neq 0$ for all eigenvectors of $L_{A^T} + \Pi$ (i.e. the system is even exactly observable). To show this, we consider M_ε^T with $F = 0$, which represen

For X

$[x_1, 0, x$

λ vec X i

of $L_{A^T} + \Pi$, if $CX = 0$.

作为应用，我们推导另外一个条件，这个条件等价于命题 8 的条件(i)-(iii). 该结论推广了张和陈（2004）的定理 6。

It is interesting that by on, our notion of detectability can be extend solvent positive operators:

Definition 13. Let $T : H$ esolvent positive, and $Y \in H$. Then the pair (T, Y) is *detectable*, if $\langle Y, V\rangle \neq 0$ for all eigenvectors V of T^* co ponding to an eigenvalue $\lambda \geqslant 0$.

As an application, we derive another condition equivalent to the conditions (i), (ii), (iii) of Proposition 8. This result extends Theorem 6 of Zhang and Chen (2004).

Proposition 14. *Let $T : H \to H$ be resolvent positive and set $\beta(T) = \max \operatorname{Re} \sigma(T)$. Then $\beta(T) < 0$ if and only if*

(iv) $\exists X, Y \in H_+ : T(X) = -Y$ *and* (T, Y) *is detectable.*

criteria to a larger class of systems, is the fact that by Theorem 3 we obtain a better understanding of these criteria.

ions:

(i) For stochastic systems, different concepts of detectability can be thought of, which unlike in the deterministic case, are not equivalent.

(ii) Detectability defined as the property that the system always produces some non-zero output if the state process is unstable, is equivalent to a generalized Hautus criterion; the latter is an appropriate algebraic criterion to deal with generalized Lyapunov and Riccati equations.

(iii) Detectability defined as the property that the dual system is stabilizable, does not have a natural interpretation. In particular, this property does not offer a method to reconstruct the state from measurements.

(iv) It is useful to view stability, stabilizability and detectability as properties of certain resolvent positive operators.

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Technical Notes and Correspondence

On Eigenvalue Sets and Convergence Rate of Itô Stochastic Systems With Markovian Switching

Zhao-Yan Li, Bin Zhou, *Member, IEEE*, Yong Wang, and Guang-Ren Duan, *Senior Member, IEEE*

Abstract—This techn ... ncerned with stability analysis and stabilization of Itô stocha ... h Markovian switching. A couple of eigenvalue sets for so ... or associated with the stochastic system under study ... erize its stability in the mean squ ... whi ... sam ... teri ... con ... termined by some eigenvalue set. Finally, a linear matrix inequality based

Bin Zhou，国家优青。Guang-Ren Duan，国家杰青，长江学者， 2 次获得国家自然科学二等奖。

past several decades, relatively little work is done for Itô stochastic system with Markovian switching. Only recently, some results for Itô stochastic system and Markovian jump systems were extended to Itô stochastic system with Markovian switching. For example, stabilization of a class of nonlinear stochastic differential equations (satisfying Lipschitz condition) with Markovian switching was considered in [17], robust stability and controllability of a class of stochastic differential delay equations with Markovian switching was solved in [18], stabilization of nonlinear and bilinear uncertain Itô stochastic systems with Markovian switching and time-delay was respectively studied in [16] and [19], robust filtering problem associated with Itô stochastic system ... was addressed in [8] and [9], sliding-mode ... sed for Itô stochastic system with Markovian ... ut-to-state stability for this class of systems with delay was considered in [5]. For more related work on this

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system, see [5], [11], [12] and the recent monographs [22]–[24].

In this technical note, motivated by the recent work [21], we consider stability analysis and stabilization problems for Itô stochastic system with Markovian switching. A series of eigenvalue s… s for some operator associated with the Markovian jumping Itô… chastic system are defined. Properties for these different eigenval… ts are proposed and, especially, it is shown that they are equival… characterizing the stability of the stochastic system under consi… n. Based on the properties of some eigenvalue set, the converge… Itô stochastic system with Markovian switching in the mean… se is analyzed, which reveals some extensions of the eigen… results for deterministic linear systems to stochastic line… e also propose an effective linear matrix inequalities (L… proach for designing feedback gains such that the cl… m has desired convergence rate. Our research in this … ds to a couple of open problems that should be caref… future.

II. PREL…

在这个技术报告中，受最近的工作[21]启发，我们考虑带Markov切换的随机Ito系统的稳定性分析和镇定问题。

[21] W. Zhang, L. Xie. Interval stability and stabilization of linear stochastic systems, IEEE Trans. Automatic Control, 54(4), pp. 810-815, 2009.

terizing the stability of the Markovian jump Itô stochastic linear system (4). Our results also generalize several aspects of those in [21]. For example, the sets $\sigma_{RS}(\mathcal{L})$, $\sigma_{\mathcal{H}}(\mathcal{L})$, $\sigma_{RS}(\mathcal{L})$ and σ_S… were not considered in that paper and Theorem 1 is totally new.

Remark 2: We notice that the results presented in this s… be extended to… jumping param… use the fact that the operator $\mathcal{L}$ is a resolvent positive operator in the sense of Definition 1. Moreover, from the above discussion, we conclude that the most convenient way for testing the stability of system (4) based on its eigenvalues is to compute the maximal real eigenvalue of $S_{\mathcal{L}}$.

我们的结论在几个方面推广了文献[21]的有关结果。

Remark 3: As we can see, the proofs for the results in this subsection do not unitize explicitly the special structures of the operator $\mathcal{L}$. In fact, some results, say, Proposition 1 and Theorem 1, hold for any resolvent positive operator. The proofs can be carried out analogously.

B. Convergence Rate Analysis

Assume that the initial condition for system (4) is $x(0) = x_0$. Then according to Lemma 4, we have $\dot{X}(t) = \mathcal{L}(X(t))$, $X(0) = (0,\cdots,0,X_{\theta_0}(0),0,\cdots,0)$, where $X_i(t)$ is defined in (5) and accordingly $X_{\theta_0}(0) = x_0x_0^T$ (here $X_{\theta_0}(0)$ is at the θ_0-th column of $X(0)$). Moreover, it is easy to see that

$$\mathbf{E}\{\|x(t)\|^2\} = \mathrm{tr}\left(\mathbf{E}\{\|x(t)\|^2\}\right) = \mathrm{tr}\left(\mathbf{E}\left\{\left(x^T(t)x(t)\right)\right\}\right) = \mathrm{tr}\left(\sum_{i=1}^{N} X_i(t)\right). \quad (23)$$

Definition 3: Let β and α be two given scalars such that $\beta \le \alpha$. The Markovian jump Itô stochastic system (4) is said to have guaranteed convergence rate $\{\beta, \alpha\}$ if for any sufficiently small scalar $\varepsilon > 0$, there exist two finite constants $c_\alpha(\varepsilon)$ and $c_\beta(\varepsilon)$ such that $c_\beta(\varepsilon)e^{(\beta-\varepsilon)t}\|x_0\|^2 \le \mathbf{E}\{\|x(t)\|^2\} \le c_\alpha(\varepsilon)e^{(\alpha-\varepsilon)t}\|x_0\|^2$.

Basically, the meaning of the guaranteed convergence rate $\{\beta, \alpha\}$ is th… (4) i… Clea… $\mu_S(\mathcal{L}) = \min\{\ldots$

这里的证明可以仿照文献[21]的定理 2.2 –(1) 的思路进行。

Theorem 2: $\beta_0 \ge \ldots$ $\rho_S(\mathcal{L}) = \rho_C(\mathcal{L})$.

Proof: We can first s… wing relation:

$$\beta_0 \ge \mu_S(\mathcal{L}), \quad \ldots \le \rho_S(\mathcal{L}). \quad (24)$$

The proof can be carried out by using the idea found in the proof of item 1 of Theorem 2.2 in [21]. The details are omitted for brevity. The remaining of this proof is to show that $\alpha_0 = \rho_S(\mathcal{L})$.

We assume that $\alpha_0 < \rho_S(\mathcal{L})$. Then there exists a sufficiently small scalar $\varepsilon > 0$ such that

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1512 IEEE/ASME TRANSACTIONS ON MECHATRONICS, VOL. 23, NO. 4, AUGUST 2018

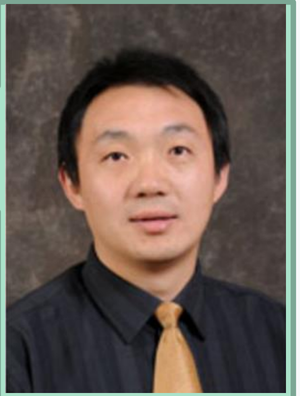
Positioning-Tracking Controller Design of A Linear Motion Control System Based on Vectorization Technique

Li Qiu, *Member, IEEE*, Yang Shi, *Fellow, IEEE*, Jianfei Pan, *Member, IEEE*, Bo Zhang, *Member, IEEE*, and Gang Xu

***Abstract*—This paper addresses the positioning-tracking control problem for a second-order direct-drive linear-switched reluctance machine (LSRM) motion control system based on the vectorization technique. In order to overcome the system-matrix dimension oversize problem caused by vectorization method, the $\mathcal{H}$-representation technique is adopted to reduce the closed-loop system-matrix dimension. The stability conditions with lower computational complexity for the LSRM motion control system are obtained based on Lyapunov stability theory and the vectorization technique. The positioning-tracking controller design method is proposed according to the matrix eigenvalue numerical-analysis method. The proposed controller design method theoretically explicitly specifies the range of the designed controller gains, which can greatly reduce the burden of setting and tuning the control parameters as compared with proportion-integral-derivative parameters tuning method. Several groups of experimental tests are presented to verify the effectiveness of the proposed positioning-tracking control method for LSRM motion control systems.**

***Index Terms*—Linear-switched reluctance machine (LSRM), positioning-tracking controller design, vectorization technique, $\mathcal{H}$-representation technique.**

minimum-relu ... iple is extended to a variety of in-
dustrial applications, ... ssembly, printed
board drilling, etc., since ... e.g., h
high precision, fast response, ...
straight-line execution, etc. Howeve...
high-precision positioning control becau...
tain the accurate mathematical model paramete...
sensitive and variations friction coefficients tractio...
In the recent years, some results have been propo...
linear-switched reluctance machines (LSRM) cont...
for example, a flux feedback reluctance actuator li...
scheme has been described based on cascaded anal...
coil and digital hall probe feedback control [2]. Th...
trol problem is studied for the discrete-time direct-...
motion control system based on Lyapunov theory in...
tematic control design method is proposed for the ...
linear motor to provide high-speed and high-precis...
mance by using fast nonsingular terminal-sliding mode in [4].
In [5], a new methodology is presented to determine the magnetically guided robot position in horizontal plane by using



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B. Stabilizability and Spectrum

The stabilization of the discrete-time closed-loop system in (8) is described by the spectral-mapping technique extended from [24]. The system in (8) can be stabilized only if there exists a constant matrix $K \in \mathbb{R}^{m\times n}$ such that the spectrum of the closed-loop operator $\mathfrak{D}_K : \xi \in \mathfrak{C}^n \longmapsto (A+BK)\xi$ satisfies $\sigma(\mathfrak{D}_K) \subset \mathfrak{C}_I$.

Definition 1: The system in (6) is stabilizable, if there exists a feedback control $u(k) = Kx(k)$, such that for any $x_0 \in \mathbb{R}^n$, the closed-loop system $x(k+1) = (A+BK)x(k)$, $x(0) = x_0$ is asymptotically stable, that is, we have

$$\begin{cases} \lim_{k\to\infty} \|x(k) - 0\| = \lim_{k\to\infty} \|A^k x_0\| = 0, \\ x_0 \neq 0, k = 0, 1, 2... \end{cases} \tag{9}$$

where $K \in \mathbb{R}^{m\times n}$ is a constant matrix.

Definition 2: By extending Definition 2 of [24], for any given feedback gain matrix K, let the generalized operator $\mathfrak{D}_A$ from $\mathfrak{S}_n$ to $\mathfrak{S}_n$ be defined as follows: $\mathfrak{D}_A : Z \in \mathfrak{S}_n \longmapsto A^{\mathrm{T}} Z A - Z \in \mathfrak{S}_n$, where the spectrum of $\mathfrak{D}_A$ is the set defined by $\sigma(\mathfrak{D}_A) := \{\lambda \in \mathfrak{C} : \mathfrak{D}_A(X) = \mathfrak{D}_A X = \lambda X, X \in \mathfrak{S}_n, X \neq 0\}$.

Proposition 1: The system in (6) is stabilizable iff there exists an $K \in \mathbb{R}^{m\times n}$, such that the spectrum of $\sigma(\mathfrak{D}_A)$ belongs to $\mathfrak{C}_I$.

Proof: It can be extended according to Theorem 1 for a continuous-time stochastic system in [24]. By Definitions 1 and 2, it suffices to prove that there exists a matrix K such that $\lim_{k\to\infty} \|\bar{A}^k x_0\| = 0, x_0 \neq 0$ is equivalent to $\sigma(\mathfrak{D}_A) \subset \mathfrak{C}_I$, where $\bar{A} = A + BK$ and $x(k)$ is the trajectory of (8). ■

Let $X(k) = \|x(k)\| = \|\bar{A}^k x_0\|$, by Lyapunov equation

$$\Delta X = \bar{A}^{\mathrm{T}} X \bar{A} - X = \mathfrak{D}_A(X)$$
$$X(0) = X_0 = \|\bar{A}^k x_0\|. \tag{10}$$

$X(\cdot)$ is real symmetric, and the system in (10) is a linear system with $n(n+1)/2$ different variables. We define a map $\widetilde{\mathfrak{D}}$ from $\mathfrak{D}_n$ to $\mathfrak{D}_{n(n+1)/2}$ as follows: For any $Z = (Z_{ij})_{n\times n} \in \mathfrak{D}_n$, set

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$$\widetilde{Z} = \widetilde{\mathfrak{D}}(Z) = (Z_{11}, \ldots, Z_{1n}, \ldots, Z_{n-1,n-1}, Z_{n-1,n}, Z_{nn})^{\mathrm{T}}$$

then, there exists a unique matrix $L(K) \in \mathbb{R}^{n(n+1)/2, n(n+1)/2}$, such that (10) is equivalent to

$$\Delta\widetilde{X} = \widetilde{\mathfrak{D}}(\mathfrak{D}_A(X)) = L(K)\widetilde{X}, \widetilde{X}(0) = \widetilde{X}_0 \quad (11)$$

where $\widetilde{X} \in \mathbb{R}^{n(n+1)/2}$ due to that $X = \|\bar{A}^k x_0\|$ is real positive semidefinite. By stability theory on difference equations, we have

$$\lim_{k\to\infty} \|\bar{A}^k x_0\| = 0, x_0 \neq 0 = \lim_{k\to\infty} \|X(k)\| = 0$$

$$\Leftrightarrow \lim_{k\to\infty} \widetilde{X}(k) = 0 \Leftrightarrow \sigma(L(K)) \subset \mathfrak{C}_I. \quad (12)$$

According to Definition 2, for any eigenvalue λ and its corresponding eigenvector $Z = (Z_{ij})_{n\times n} \in \mathfrak{D}_n$ of $\mathfrak{D}_n$, from $\mathfrak{D}_n(Z) = \lambda Z$, we have $L(K)\widetilde{Z} = \lambda\widetilde{Z}$, which yields $\sigma(L(K)) = \sigma(\mathfrak{D}_n)$. The above completes the proof of Proposition 1.

C. $\mathcal{H}$-Representation Matrixes

Definition 3: [25] Consider a $p-$ dimensional complex (real) matrix subspace $\mathbb{X} \subset \mathcal{C}^{n\times n}$ ($\mathbb{X} \subset \mathcal{R}^{n\times n}$) over the fields $\mathcal{C}$ ($\mathcal{R}$). Assume that $e_1, e_2, \ldots e_p$ form the basis of $\mathbb{X}$, and define $H = [\overrightarrow{e_1} \quad \overrightarrow{e_2} \quad \cdots \quad \overrightarrow{e_p}]$. For each $X \in \mathbb{X}$, if we express vec $(X) = \overrightarrow{X}$ in the form of

$$\mathrm{vec}(X) = \overrightarrow{X} = H\widetilde{X}$$

with a $p \times 1$ vector $\widetilde{X}$, the $H\widetilde{X}$ is called an $\mathcal{H}$-representation of vec(x), and H is called an $\mathcal{H}$-representation matrix of vec(X).

Lemma 1: [25] H has full column rank, namely, $[H^{\mathrm{T}}H]$ is invertible.

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Delay-Dependent Algebraic Riccati Equation to Stabilization of Networked Control Systems: Continuous-Time Case

Cheng Tan, Huanshui Zhang, *Senior Member, IEEE*, and Wing Shing Wong, *Fellow, IEEE*

***Abstract*—In this paper, a delay-dependent algebraic Riccati equation (DARE) approach is developed to study the mean-square stabilization problem for continuous-time networked control systems. Different from most previous studies that information transmission can be performed with zero delay and infinite precision, this paper presents a basic constraint that the designed control signal is transmitted over a delayed communication channel, where signal attenuation and transmission delay occur simultaneously. The innovative contributions of this paper are threefold. First, we propose a necessary and sufficient stabilizing condition in terms of a unique positive definite solution to a DARE with $Q > 0$ and $R > 0$. In accordance with this result, we derive the Lyapunov/spectrum stabilizing criterion. Second, we apply the operator spectrum theory to study the stabilizing solution to a more general DARE with $Q \geq 0$ and $R > 0$. By defining a delay-dependent Lyapunov operator, we propose the existence theorem of the unique stabilizing solution. It is shown that the stabilizing solution, if it exists, is unique and coincides with a maximal solution. Third, as an application, we derive the explicit maximal allowable delay bound for a scalar system.**

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Due to the insertion of digital co… with finite capacity and resource, many … cannot be satisfied. Some examples … separation of estimation and control, … and synchroneity among multiple a… transmission delay often occurs while e… from different devices connected to a sh… Moreover, control signal often suffers signal attenuation over fading channels [8], [9]. These uncertainties may degrade the quality of network service and even destabilize some time-sensitive system. As a consequence, it is of significance to study the stabilization problem for NCSs over delayed

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In the rest of this section, we propose the spectrum stabilizing criterion. Define the following delay-dependent Lyapunov operator $\mathcal{L}_K(\cdot)$ from $\mathbb{S}^n$ to $\mathbb{S}^n$:

$$\mathcal{L}_K(X) \triangleq (A+\mu BK)X + X(A+\mu BK)' + \sigma^2 e^{Ad} BKXK'B'e^{A'd}, \quad \forall X \in \mathbb{S}^n. \tag{29}$$

Definition 1: The spectrum set of operator $\mathcal{L}_K(\cdot)$ is defined as

$$\sigma(\mathcal{L}_K) \triangleq \{\lambda \in \mathbb{C} : \mathcal{L}_K(\hat{X}) = \lambda \hat{X}, \hat{X} \in \mathbb{S}^n, \hat{X} \neq 0\} \tag{30}$$

where $\hat{X}$ is said to be an eigenvector associated with the eigenvalue λ.

With the inner product, defined by $\langle U, V\rangle = \mathrm{Tr}(UV)$ for any $U, V \in \mathbb{S}^n$, the adjoint Lyapunov operator $\mathcal{L}_K^*(\cdot)$ from $\mathbb{S}^n$ to $\mathbb{S}^n$ is given as

$$\mathcal{L}_K^*(X) \triangleq (A+\mu BK)'X + X(A+\mu BK) + \sigma^2 K'B'e^{A'd}Xe^{Ad}BK, \quad \forall X \in \mathbb{S}^n. \tag{31}$$

Then, for any $X, Y \in \mathbb{S}^n$, we have

$$\mathrm{Tr}(X\mathcal{L}_K(Y)) = \mathrm{Tr}(\mathcal{L}_K^*(X)Y). \tag{32}$$

It follows from [23] that $\sigma(\mathcal{L}_K) = \sigma(\mathcal{L}_K^*)$. Now, we are in a position to derive a spectral stabilizing description for the NCS, $[A, B|d, \xi]$.

equivalent to the feasibility of a certain LMI. Moreover, by introducing a delay-dependent Lyapunov operator, we demonstrate that the NCS is stabilizable in the mean-square sense if and only if the operator spectrum set belongs to the open left-hand side of the complex plane. Note that these stabilizing criteria are first obtained in the framework of continuous-time stochastic system with both input delay and multiplicative noises, which run in parallel to the classical stochastic results in [15] and [23]. Second, we investigate the solvability of a

Corollary 1: The NCS, $[A, B|d, \xi]$, is stabilizable in the mean-square sense if and only if there exists a feedback gain matrix K such that the spectrum of $\mathcal{L}_K(\cdot)$ satisfying

$$\sigma(\mathcal{L}_K) \subseteq \mathbb{C}^- \triangleq \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}. \tag{33}$$

Proof: The proof can be derived directly by applying [23, Th. 1] and Theorem 2. ■

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Necessary and Sufficient Stabilizing Conditions for Networked Control Systems With Simultaneous Transmission Delay and Packet Dropout

Cheng Tan and Huanshui Zhang

***Abstract*—This paper investigates the mean-square stabilization problem for discrete-time networked control systems (NCSs). Different from most previous studies, we assume transmission delay and data packet dropout may occur simultaneously. The stabilization for such NCSs remains challenging because of the fundamental difficulty in stochastic control. The contributions of this paper are threefold. First, we present two different necessary and sufficient stabilizing conditions in terms of the unique positive solution to delay-dependent algebraic Riccati equation (DARE) or delay-dependent Lyapunov equation (DLE). Second, the maximum packet dropout rate can be calculated with a proposed optimization algorithm. Third, the stabilizing solution to developed DARE is investigated for its existence and uniqueness. We show the existence condition in terms of the Lyapunov operator and the unobservable mean-square eigenvalue, under which the general DARE has a unique stabilizing solution.**

***Index Terms*—DARE, networked control system (NCS), packet dropout, stabilization, stabilizing solution.**

Recently, some studies have concentrated on the simultaneous occurrence of these two uncertainties [7]–[9]. The most popular methods, including the switched approach and Lyapunov-Krasivskii functional approach, are mainly on LMIs, and only sufficient stabilizing conditions are available. Besides, except for some special cases investigated in [10], how to derive the explicit value of the maximum packet dropout rate of the general system is to be resolved. As said in [7], [8], the stabilization problems for the NCSs with simultaneous transmission delay and packet dropout are challenging, difficult and unsolved. This can be ascribed to the fundamental difficulty of stochastic control, i.e., the separation principle fails for a stochastic system with input delay and multiplicative noises.

In this paper, we focus on the mean-square stabilization problem for discrete-time NCSs with simultaneous transmission delay and packet dropout. First, motivated by the coupled Riccati-ZXL equation in our

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2) For system $(A; Q^{\frac{1}{2}})$, the mean-square eigenvalue $\lambda \in \sigma_{\mathbb{S}^{n+}}(\mathcal{L}_A)$ is said to be an observable mean-square eigenvalue, if its corresponding nonzero eigenvector $X \in \mathbb{S}^{n+}$ satisfying $\mathcal{L}_A(X) = \lambda X$ and $Q^{\frac{1}{2}} X \neq 0$; $\lambda \in \sigma_{\mathbb{S}^{n+}}(\mathcal{L}_A)$ is said to be an unobservable mean-square eigenvalue, if its corresponding nonzero eigenvector $X \in \mathbb{S}^{n+}$ satisfying $\mathcal{L}_A(X) = \lambda X$ and $Q^{\frac{1}{2}} X = 0$.

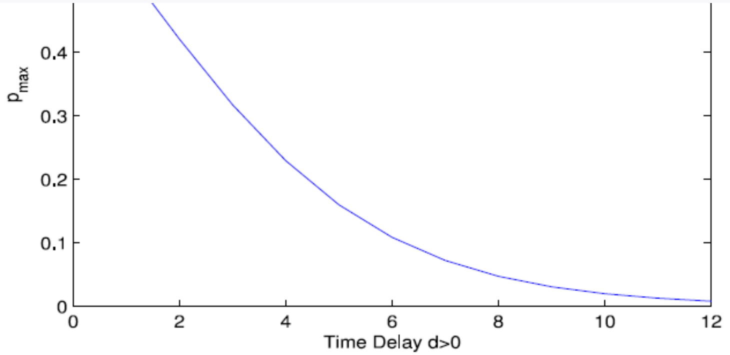
Lemma 2: (PBH Criterion)

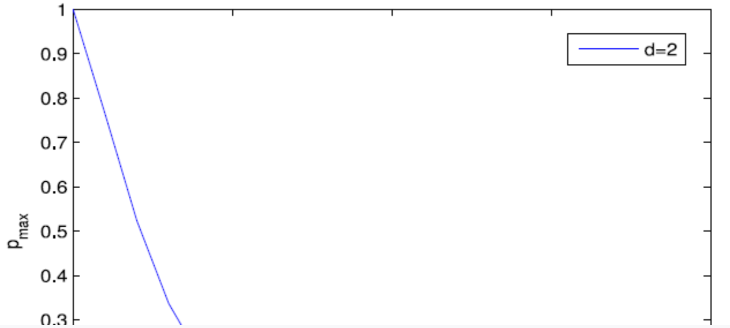
1) System $(A; Q^{\frac{1}{2}})$ is observable iff there does not exist any unobservable mean-square eigenvalue.

2) System $(A; Q^{\frac{1}{2}})$ is detectable iff any unstable mean-square eigenvalue $\lambda \in \sigma_{\mathbb{S}^{n+}}(\mathcal{L}_A)$ is observable.

Proof: This proof follows from Theorem 3.1 in [16] and Theorem 4 in [17] and thus omitted. ■

Proposition 1: Assume $P \geq 0$ is a positive-semidefinite solution to the DARE (14) with $Q \geq 0$ and $R > 0$. Then, any unstable mean-square eigenvalue $\lambda \in \sigma_{\mathbb{S}^{n+}}(\mathcal{L}_{K_P})$ of system $[A, B, d|p]$ with $K_P = -\Upsilon^{-1} M$ is the unobservable mean-square eigenvalue of system $(A; Q^{\frac{1}{2}})$.





[21] W. Zhang, B.S.Chen. Automatica, 40(1), 87-94,2004 , 2004.

Thank you for your attendance!