Cooperative Game

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Outline of Presentation

I. Cooperative Game and Characteristic Function

☞ What is a Coordinative Game?

Definition 1.1

A cooperative game is determined by a couple $(N,\ v)$, where

(i) $N = \{1, 2, \cdots, n\}$ is the set of players(玩家);

(ii) $v : 2^N \to \mathbb{R}$ is a section mapping, satisfying $v(\phi) = 0$, called a characteristic function(特征函数).

A subset of N, denoted by $S \subset N$, (or $S \in 2^N$), is called a colleague(联盟). $v(S)$ represents the value of this colleague. The main purpose of Cooperative Game Theory is to provide a fair rule, which determines the payments of individual players. This rule is called an imputation(分配).

☞ Some Examples

Example 1.2

(Gloves) There are N persons, every player has a single glove. Assume R: the set of persons who have right gloves; and L: the set of persons, who have left gloves. A pair of gloves is worth $ 2, and a single glove is worth $0.02. Find the characteristic function?

Let $S \subset N$. Then

- Number of pairs:

$$N_P = \min(|R \cap S|, |L \cap S|).$$

- Number of the remaining single gloves:

$$N_s = |S| - 2N_P.$$

Hence,

$$v(S) = 2 \times N_P + 0.02 \times N_s.$$

Example 1.3

(Selling Horse) A person (A) is going to sell a horse, the minimum price he asked is $\$100$. Two persons (B and C) want to buy a house, the price B is willing to pay is $\$100$, C is $\$110$. Calculating the characteristic function.

In this game, $N = \{A, B, C\}$.

$$2^N = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\} .$$

By definition,

$$v(\emptyset) = 0.$$

Example 1.3(cont'd)

If there is no trade, the characteristic function equals to 0. Hence

$$v(\{A\}) = v(\{B\}) = v(\{C\}) = v(\{B, C\}) = 0.$$

If there is a trade, the we have the following:

$$v(\{A, B\}) = 100; v(\{A, C\}) = 110.$$

Similarly,

$$v(\{A, B, C\}) = 110.$$

Example 1.3(cont'd)

We conclude that

$$v(S) = \begin{cases} 110, & S = \{A, B, C\}, \\ 100, & S = \{A, B\}, \\ 110, & S = \{A, C\}, \\ 0, & S = \{A\}, \\ 0, & S = \{B\}, \\ 0, & S = \{C\}, \\ 0, & S = \emptyset. \end{cases} \tag{1}$$

Example 1.4

一位导师(T) 带两个学生(A) 和(B)。A 做理论研究, B 做实验。如果B 单独工作, 写不出论文; A 单独工作, 可写一篇核心论文, 值 1 个单元; A 与B 合作或老师单独工作, 均可写出一篇SCI 四区论文, 值 2 个单元; 如果老师与B 合作, 可写出一篇SCI 三区论文, 值 4 个单元; 如果老师与A 合作, 可写出一篇SCI 二区论文, 值 7 个单元; 如果老师与A, B 共同合作, 可写出一篇SCI 一区论文, 值 10 个单元. 那么, $G = \{N = \{T, A, B\}, v\}$, 这里

$$\begin{array}{ll} v(\emptyset) = 0; & v(B) = 0; \\ v(A) = 1; & v(A \cup B) = 2; \\ v(T) = 2; & v(T \cup B) = 4; \\ v(T \cup A) = 6; & v(T \cup A \cup B) = 10. \end{array} \tag{2}$$

Example 1.4(cont'd)

于是有

$$v(S) = \begin{cases} = 0; & S = \emptyset, \text{ or } \{B\}, \\ = 1, & S = \{A\}, \\ = 2, & S = \{T\}, \text{ or } \{A, B\}, \\ = 4, & S = \{T, B\}, \\ = 6, & S = \{T, A\}, \\ = 10, & S = \{T, A, B\}. \end{cases} \tag{3}$$

☞ Vector Form of Characteristic Function

Let $S \in 2^N$. It can be expressed by an index function $I_s \in \mathcal{D}^n$. Denote $I_S = (s_1, s_2, \cdots, s_n)$, where

$$s_j = \begin{cases} 1, & j \in S \\ 0, & j \notin S. \end{cases}$$

Since $s_i \in \mathcal{D} = \{0, 1\}$, $i = 1, 2, \cdots, n\}$, then a characteristic function υ can be considered as a pseudo-Boolean function

$$\upsilon(S) = \upsilon(s_1, s_2, \cdots, s_n) : \mathcal{D}^n \to \mathbb{R}. \tag{4}$$

☞ Algebraic Representation of Characteristic Function

Setting $1 \sim \delta_2^1$, $0 \sim \delta_2^2$, then $s_j \in \Delta_2$, $j = 1, 2, \cdots, n$. For each characteristic function υ, there is a structure vector denoted by V_υ, such that

$$\upsilon(S) = V_\upsilon \ltimes_{i=1}^n s_i. \quad (5)$$

Note that $V_\upsilon \in \mathbb{R}^{2^n}$, and $\upsilon(\phi) = 0$, the last component of V_υ is 0. Hence,

Proposition 1.5

Let $|N| = n$, Then the set of cooperative games over N, denoted by $G(N)$, form a $2^n - 1$ dimensional vector space, which is isomorphic to $\mathbb{R}^{2^n-1}$.

☞ Essential/Non-Essential Game

Definition 1.6

Consider (N, v).

(i) v is said to satisfy super-additivity (超可加性) if for any two colleagues $P,\ Q \in 2^N$ and $P \cap Q = \emptyset$:

$$v(P \cup Q) \geq v(P) + v(Q). \tag{6}$$

(N, v) is called an essential game (本质博弈) if $>$ holds for some $(R,\ S)$.

(ii) v is said to satisfy additivity (可加性) if for any two colleagues $P,\ Q \in 2^N$ and $P \cap Q = \emptyset$:

$$v(P \cup Q) = v(P) + v(Q), \tag{7}$$

(N, v) is called a non-essential game (非本质博弈).

Theorem 1.7

(N, v) is a non-essential game, if and only if,

$$v(N) = \sum_{i=1}^{n} v(i). \tag{8}$$

Definition 1.8

(N, v) is an essential game if

$$v(N) > \sum_{i=1}^{n} v(i).$$

We are only interested in essential games!

II. Zero-Sum (Constant-Sum) Game

☞ What is a Zero-Sum Game

Definition 2.1

A constant-sum game is a game $G = (N, S, C)$. If

$$\sum_{i=1}^{n} c_i(x_1, x_2, \cdots, x_n) = \mu, \quad x_i \in S_i, \quad \forall i. \tag{9}$$

If $\mu = 0$, G is a zero-sum game.

Example 2.2

Zero-sum game:

(i) Rock-Paper-Scissors(石头-剪刀-布),

(ii) Tienji Horse Racing(田忌赛马)

(iii) Palm-up Palm-down (手心手背)

☞ Two Player Zero-Sum Game

Consider $G \in \mathcal{G}_{2;p,q}$:

Payoff Matrix

$$A_1 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,q} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,q} \\ \vdots & & & \\ a_{p,1} & a_{p,2} & \cdots & a_{p,q} \end{bmatrix}$$

$$A_2 = -A_1.$$

Proposition 2.2

Assume $G \in \mathcal{G}_{[2,p,q]}$. Then

(i)

$$\max_{1\le i\le p}\min_{1\le j\le q} a_{i,j} \le \min_{1\le j\le q}\max_{1\le i\le p} a_{i,j}. \tag{10}$$

(ii) The necessary and sufficient condition for

$$\max_{1\le i\le p}\min_{1\le j\le q} a_{i,j} = \min_{1\le j\le q}\max_{1\le i\le p} a_{i,j}, \tag{11}$$

is there exists (i^*, j^*) such that

$$a_{i,j^*} \le a_{i^*,j^*} \le a_{i^*,j}, \quad i = 1,2,\cdots,m;\ j = 1,2,\cdots,n. \tag{12}$$

Proposition 2.2(cont'd)

(iii) For mixed strategies, there exists at least one (x^*, y^*) such that

$$\max_{x \in \bar{S}_1} \min_{y \in \bar{S}_2} E(x, y) = \min_{y \in \bar{S}_2} \max_{x \in \bar{S}_1} E(x, y) = E(x^*, y^*). \quad (13)$$

Note that (x^*, y^*) is a Nash equilibrium.

Proposition 2.3

Let (x^*, y^*) and $(\bar{x}, \bar{y})$ be two Nash equilibria of a two player zero-sum game. Then

$$Ec_1(x^*, y^*) = -Ec_2(x^*, y^*) = Ec_1(\bar{x}, \bar{y}) = -Ec_2(\bar{x}, \bar{y}) \quad (14)$$

☞ n Player Zero-Sum Game

Consider n player zero-sum game. Let $R \subset N$ and and $R^c \neq \emptyset$. To evaluate of value of R, it is natural to define it as its payoff in fighting with R^c.
The strategies for R and R^c are:

$$S_R = \prod_{i \in R} S_i, \quad S_{R^c} = \prod_{i \in R^c} S_i.$$

The game between R and R^c becomes a two player zero sum game. Then we can define

$$\begin{aligned} \upsilon(R) &:= \max_{\xi \in \bar{S}_R} \min_{\eta \in \bar{S}_{R^c}} \sum_{r \in R} e_r(\xi, \eta) \\ &= \min_{\eta \in \bar{S}_{R^c}} \max_{\xi \in \bar{S}_R} \sum_{r \in R} e_r(\xi, \eta) \\ &= \sum_{r \in R} e_r(\xi^*, \eta^*), \end{aligned} \tag{15}$$

where (ξ^*, η^*) is a Nash equilibrium of the game over (R, R^c).

Define

$$v(\emptyset) = 0,$$
$$v(N) = \max_{s \in S} \sum_{i=1}^{n} c_i(s). \tag{16}$$

Then (N, v) becomes a cooperative game.

☞ My Homework

Example 2.a

A boy and a girl play matching penny: The payoff bi-matrix is

表 1: Payoffs for Example 2.a

$B \backslash G$	H	T
H	$3,\ -3$	$-2,\ 2$
T	$-2,\ 2$	$1,\ -1$

Example 2.a(cont'd)

The Nash equilibrium is: $p^* = (3/8, 5/8), q^* = (3/8, 5/8)$.

Consider it as a cooperative game. Using (15), we have

$$\begin{array}{rcl} v(\emptyset) & = & 0, \\ v(\{B\}) & = & \frac{3}{8}\frac{3}{8} * 3 + \frac{3}{8}\frac{5}{8} * (-2) \\ & & +\frac{5}{8}\frac{3}{8} * (-2) + \frac{5}{8}\frac{5}{8} * (1) = -\frac{1}{8}, \\ v(\{G\} & = & -\frac{1}{8}, \\ v(\{B,\ G\} & = & 0. \end{array}$$

A constant sum non-cooperative game has a naturel cooperative game structure!

Remark 2.4

For non-constant game, is it possible to use

$$v(R) := \max_{\xi \in \bar{S}_R} \min_{\eta \in \bar{S}_{R^c}} e_R(\xi, \eta) \tag{17}$$

or

$$v(N) = \max_{s \in S} \sum_{i \in N} c_i(s)$$

to define characteristic function?

Main problem: super-additivity is not ensured!

☞ Properties of Characteristic Function of Zero-Sum Games

Proposition 2.5

Let v be the characteristic function of zero-sum games (defined as above). Then

$$v(R) + v(R^c) = v(N), \quad \forall R \in 2^N. \tag{18}$$

Proposition 2.6

Let v be the characteristic function of zero-sum games. Then (super-additivity)

$$v(S \cup T) \geq v(S) + v(T). \tag{19}$$

Remark 2.7

- No possible cooperation in zero-sum game with 2 players.
- There is a possibility for cooperation in zero-sum game with more than 2 players.

Example 2.8

A palm-up palm-down game with three players are considered. Denote by $S_1 = S_2 = S_3 := S_0 = \{1, 2\}$, where

$$1: \text{ palm-up}; \quad 2: \text{ palm-down}.$$

Example 2.8(cont'd)

The payoff matrix is shown in Table 2.

表 **2:** Payoffs for Example 2.8

$c\backslash p$	111	112	121	122	211	212	221	222
c_1	0	1	1	−2	−2	1	1	0
c_2	0	1	−2	1	1	−2	1	0
c_3	0	−2	1	1	1	1	−2	0

Example 2.8(cont'd)

We may consider the best payoffs as the characteristic function.

(i) Since the game is zero-sum, we have $v(1,2,3) = 0$.

(ii) Consider $v(1,2)$. Take $R = \{1,2\}$ as one side, $R^c = \{3\}$ as the other side, then the payoff matrix of R can be expressed as in Table 3.

表 **3:** Payoff of R vc R^c

$R = \{1,2\} \backslash R^c = \{3\}$	1	2
11	0	2
12	-1	-1
21	-1	-1
22	2	0

Example 2.3(cont'd)

No matter what strategy 3 chosen, for R 12 or 21 is wroth than 11 or 22. So, row 2 and row 3 can be deleted. Hence, both R and R^c have two strategies, Denote $p = P(R = 11)$, $q = P(R^c = 1)$. then the expected value of R is

$$ER = p(1-q) \times 2 + (1-p)q \times 2.$$

Similarly,

$$ER^c = p(1-q) \times (-2) + (1-p)q \times (-2).$$

Hence, the Nash equilibrium can be calculated as

$$p^* = (1/2, 1/2) \quad q^* = (1/2, 1/2).$$

It follows from (15) that $ER = 1$, $ER^c = -1$.

Example 2.3(cont'd)

So we define

$$v(\{1,2\}) = 1, \quad v(\{3\}) = -1.$$

Because of symmetry, the vector form of characteristic function v is

$$V_v = [0, 1, 1, 1, -1, -1, -1, 0].$$

III. Two Standard Cooperative Games

☞ Unanimity Game (无异议博弈)

Definition 3.1

$G = (N, v)$ is called a unanimity game, if there exists a $\emptyset \neq T \in 2^N$, such that

$$v_T(S) = \begin{cases} 1, & T \subset S \\ 0, & \text{Otherwise.} \end{cases} \qquad (20)$$

Denote by $\mathcal{G}_n^c$ the set of cooperative games with n players. Then each $G \in \mathcal{G}_n^c$ is uniquely determined by v. Since $v(\emptyset) = 0$,

$$\mathcal{G}_n^c \sim \mathbb{R}^{2^n - 1}. \qquad (21)$$

Theorem 3.2

(i) The set of unanimity Games

$$\left\{ \upsilon_T \middle| \emptyset \neq T \in 2^N \right\},$$

form a basis of $\mathcal{G}_n^c$.

(ii) Let $\upsilon \in G^N$. Then

$$\upsilon = \sum_{T \in 2^N \setminus \emptyset} \mu_T \upsilon_T, \tag{22}$$

where

$$\mu_T = \sum_{S \subset T} (-1)^{(|T|-|S|)} \upsilon(S). \tag{23}$$

Example 3.3

Consider $G = (N = \{1, 2\}, \upsilon)$. We have subsets 2^N as:

$$S_1 = \{1, 2\}, \;\; S_2 = \{1\}, \;\; S_3 = \{2\}, \;\; S_4 = \emptyset.$$

By Definition 3.1, we have

$$\begin{array}{llll} u_{S_1}(S_1) = 1, & u_{S_1}(S_2) = 0, & u_{S_1}(S_3) = 0, & u_{S_1}(S_4) = 0, \\ u_{S_2}(S_1) = 1, & u_{S_2}(S_2) = 1, & u_{S_2}(S_3) = 0, & u_{S_2}(S_4) = 0, \\ u_{S_3}(S_1) = 1, & u_{S_3}(S_2) = 0, & u_{S_3}(S_3) = 1, & u_{S_3}(S_4) = 0, \end{array}$$

According to (22) and (23), we have

$$\upsilon = \mu_{S_1}\upsilon_{S_1} + \mu_{S_2}\upsilon_{S_2} + \mu_{S_3}\upsilon_{S_3},$$

where μ_{S_i} can be calculated by (23) as:

Example 3.3(cont'd)

$$\mu_{S_1} = \sum_{S \subset S_1} (-1)^{(|S_1|-|S|)} \upsilon(S) = v(S_1) - v(S_2) - v(S_3),$$
$$\mu_{S_2} = \sum_{S \subset S_2} (-1)^{(|S_2|-|S|)} \upsilon(S) = v(S_2),$$
$$\mu_{S_3} = \sum_{S \subset S_3} (-1)^{(|S_3|-|S|)} \upsilon(S) = \upsilon(S_3).$$

It follows that

$$\upsilon = [\upsilon(S_1) - \upsilon(S_2) - \upsilon(S_3)]\, \upsilon_{S_1} + \upsilon(S_2)\upsilon_{S_2} + \upsilon(S_3)\upsilon_{S_3}. \quad (24)$$

☞ Matrix Form of (23)

Formally set:

$$v_\emptyset(S) := \begin{cases} 1, & S = \emptyset \\ 0, & \text{Otherwise.} \end{cases}$$

And we fix

$$\mu_\emptyset = 0.$$

The formula (22) can be written as

$$v = \sum_{T \in 2^N} \mu_T v_T. \tag{25}$$

Using structure vectors V_T, V_S to express υ_T we have the following:

(i) $\|N\| = 1$:

表 **4:** υ_T for $|N| = 1$

$V_T \backslash V_S$	1	0
1	1	0
0	1	1

(ii) $\|N\| = 2$:

表 **5:** $|N| = 2$ 时的 υ_T for $|N| = 2$

$V_T \backslash V_S$	1 1	1 0	0 1	0 0
1 1	1	0	0	0
1 0	1	1	0	0
0 1	1	0	1	0
0 0	1	1	1	1

(iii) $\|N\| = 3$:

表 **6:** υ_T for $|N| = 3$

$V_T \backslash V_S$	1 1 1	1 1 0	1 0 1	1 0 0	0 1 1	0 1 0	0 0 1	0 0 0
1 1 1	1	0	0	0	0	0	0	0
1 1 0	1	1	0	0	0	0	0	0
1 0 1	1	0	1	0	0	0	0	0
1 0 0	1	1	1	1	0	0	0	0
0 1 1	1	0	0	0	1	0	0	0
0 1 0	1	1	0	0	1	1	0	0
0 0 1	1	0	1	0	1	0	1	0
0 0 0	1	1	1	1	1	1	1	1

The υ_T in above tables, denoted by U_n, is called n-th degree unanimity game, where $n = |N|$, $U_u \in \mathcal{B}_{2^n \times 2^n}$.

Proposition 3.4

The unanimity matrices can be constructed recursively as follows:

$$\begin{cases} U_1 & = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ U_{k+1} & = \begin{bmatrix} U_k & 0 \\ U_k & U_k \end{bmatrix}, \quad k = 2, 3, \cdots . \end{cases} \tag{26}$$

Theorem 3.5

The structure vector of υ satisfies

$$V_\upsilon = (\mu_1\ \mu_2\ \cdots\ \mu_{2^n})\, U_n. \tag{27}$$

Hence, the coefficients of expansion (22) satisfy

$$(\mu_1\ \mu_2\ \cdots\ \mu_{2^n}) = V_\upsilon U_n^{-1}, \tag{28}$$

where

$$\begin{cases} U_1^{-1} & = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ U_{k+1}^{-1} & = \begin{bmatrix} U_k^{-1} & 0 \\ -U_k^{-1} & U_k^{-1} \end{bmatrix}, \quad k = 2, 3, \cdots . \end{cases} \tag{29}$$

Example 3.6

Recall Example 3.3. Let $n = 2$. Using formula (27), we have

$$(v(S_1)\ v(S_2)\ v(S_3)\ 0) = (\mu_1\ \mu_2\ \mu_3\ \mu_4)\, U_2.$$

Hence,

$$\begin{aligned}
(\mu_1\ \mu_2\ \mu_3\ \mu_4) &= (v(S_1)\ v(S_2)\ v(S_3)\ 0)\, U_2^{-1} \\
&= (v(S_1)\ v(S_2)\ v(S_3)\ 0) \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \\
&= \left(v(S_1) - v(S_2) - v(S_3)\ \ v(S_2)\ \ v(S_3)\ \ 0\right).
\end{aligned}$$

☞ Equivalence of Characteristic Functions

Definition 3.7

Let (N, v) and (N, v') be two cooperative games. The characteristic functions v and v' are said to be strategically equivalent (策略等价), denoted by $v \sim v'$, if there exist $\alpha > 0$, $\beta_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, $(n = |N|)$, such that

$$v'(R) = \alpha v(R) + \sum_{i \in R} \beta_i, \quad \forall R \in 2^N. \tag{30}$$

Proposition 3.8

Assume v satisfies super-additivity, and $v \sim v'$, then v' also satisfies super-additivity.

☞ Normal Game (规范博弈)

Definition 3.9

A cooperative game is said to be a $(0,1)$-normal game (规范博弈), if it satisfies

(i) $v(\{i\}) = 0, \quad \forall i \in N;$

(ii) $v(N) = 1.$

Proposition 3.10

A cooperative game $G = (N, v)$, satisfying super-additivity, is strategy equivalent to a unique $(0,1)$-normal game.

☞ Verifying Normal Form

Since

$$v(N) - \sum_{i=1}^{n} v(\{i\}) > 0.$$

Set

$$\alpha = \frac{1}{v(N) - \sum_{i=1}^{n} v(\{i\})} > 0;$$

$$\beta_i = -\alpha v(\{i\}), \quad i = 1, 2, \cdots, n.$$

Define

$$v'(R) = \alpha v(R) + \sum_{i \in R} \beta_i, \quad \forall R \in 2^N.$$

It is easy to see that v' is $(0, 1)$-normal game.

☞ Non-Essential Game (非本质博弈)

Definition 3.11

(N, v) is called a 0-normal game (零规范博弈), if

$$v(R) = 0, \quad \forall R \in 2^N.$$

Consider a non-essential game (N, v), we have

$$v(R) = \sum_{i \in R} v(\{i\}), \quad \forall R \in 2^N.$$

Let $\alpha = 1$, $\beta_i = -v(\{i\})$. Define

$$v'(R) = v(R) - \sum_{i \in R} v(\{i\}).$$

We have $v'(R) = 0, \forall R \in 2^N$.

Proposition 3.12

Every non-essential game is equivalent to a 0-normal game.

IV. Imputation-Solution to Cooperative Game

☞ Imputation (分配)

Definition 4.1

Consider a cooperative game (N, v), an n dimensional vector $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ is called an imputation, if it satisfies

(i) Individual Rationality (个体合理性):

$$x_i \geq v(\{i\});$$

(ii) Group Rationality)(群体合理性):

$$\sum_{i=1}^{n} x_i = v(N).$$

Remark 4.2

(i) Individual Rationality ensures the payoff of each person is no lesser that "non-cooperative" case. Group Rationality ensures that all income has been distributed, and no blank cheque.

(ii) The "solution" for an cooperative game is a (reasonable) imputation.

Proposition 4.3

Non-essential game has only one imputation, which is:

$$x_i = v(\{i\}), \quad i = 1, 2, \cdots, n. \tag{31}$$

☞ Core(核心)

Proposition 4.3

The set of imputations of an essential game is an n dimensional non-empty convex set, denoted by $E(v)$.

Definition 4.4

Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be two imputations. x is said to dominate (优超) y, if there exists a $\emptyset \neq R \subset N$, such that

(i)

$$x_i > y_i, \quad i \in \mathbb{R}. \tag{32}$$

(ii)

$$v(R) \geq \sum_{i \in R} x_i. \tag{33}$$

Definition 4.5

Given a cooperative game (N, v), the set of imputations, which can not be dominated by any imputation, is called the core(核心), denoted by $C(v)$.

Theorem 4.6

Given a a cooperative game (N, v) with $|N| = n$, and $x \in \mathbb{R}^n$. $x \in C(v)$, if and only if,

(i)

$$x(R) \geq v(R), \quad \forall R \subset N. \tag{34}$$

(ii)

$$x(N) = v(N). \tag{35}$$

(Necessity needs super-additivity of v.)

☞ Numerical Method

(i) Constructing M_n:
Convert $2^n - 1, 2^n - 2, \cdots, 1, 0$ into binary forms as

$$\begin{array}{lll} b_1 = (1, 1, \cdots, 1, 1) & b_2 = (1, 1, \cdots, 1, 0) & \cdots \\ \cdots & b_{2^n-1} = (0, 0, \cdots, 0, 1) & b_{2^n} = (0, 0, \cdots \end{array}$$

Construct

$$M_n = [b_1^T, b_2^T, \cdots, b_{2^n}^T]. \quad (36)$$

(ii) Constructing N_n:
Deleting first and last columns of M_n yields $\ddot{M}_n$. Set

$$N_n = \ddot{M}_n^T. \quad (37)$$

(iii) Constructing W_υ:
Delete first and last elements of V_υ to get $\ddot{V}_\upsilon$. Define

$$W_\upsilon = \ddot{V}_\upsilon^T. \tag{38}$$

(iv) Construct a set of equality-inequality as

$$\begin{cases} \sum_{i=1}^{n} x_i = \upsilon(N), \\ N_n x \geq W_\upsilon. \end{cases} \tag{39}$$

Proposition 4.7

Consider (N, v). $x \in C(v)$, if and only if, x satisfies (39).

Example 4.8

Recall Example 1.3(Selling Horse) We have

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

$$V_v = [110, 100, 110, 0, 0, 0, 0, 0].$$

Then (39) becomes

Example 4.8(cont'd)

$$\begin{cases} x_1 + x_2 + x_3 = 110 \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 100 \\ 110 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{cases}. \tag{40}$$

Example 4.8(cont'd)

The solution is:

$$\begin{cases} x_1 \in [100,\ 110] \\ x_2 = 0 \\ x_3 = 110 - x_1. \end{cases}$$

We conclude that

$$C(v) = \{(t, 0, 110 - t) \mid 100 \leq t \leq 110\}.$$

Remark 4.9

For a given $G = (N, v) \in \mathcal{G}_n$, the corner $C(v)$ may not exist!

V. Shapley Value

☞ Permutation Group $\mathbf{S}_n$

Definition 5.1

(i) A permutation:

$$\sigma : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$$

(ii) The set of permutations:

$$\mathbf{S}_n = \{\Sigma \mid \Sigma : \mathcal{D}_n \to \mathcal{D}_n\}.$$

(iii)

$$T_\sigma^i = \{j \mid \sigma_j < \sigma_i\}.$$

Example 5.2

(i)

$$\sigma = (1,3,4)(2,5) \in \mathbf{S}_5.$$

(ii) Consider σ, then

$$\sigma(1) = 3, \quad \sigma(2) = 5, \quad \sigma(3) = 4,$$
$$\sigma(4) = 1, \quad \sigma(5) = 2.$$

It is easy to see that:

$$T_\sigma^3 = \{1,4,5\},$$
$$T_\sigma^5 = \{4\}.$$

Definition 5.3

Consider $G = (N, v) \in \mathcal{G}_n$. Define

$$\varphi_i(v) := \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \left[v\left(T_\sigma^i \cup \{i\}\right) - v\left(T_\sigma^i\right) \right], \quad i = 1, 2, \cdots, n. \tag{41}$$

Then

$$\varphi := (\varphi_1, \varphi_2, \cdots, \varphi_n) \in E(v)$$

is called a Shapley value.

Proposition 5.4

$$\sum_{i=1}^{n} \varphi_i(v) = v(N). \tag{42}$$

$$\varphi_i(v) \geq v(\{i\}). \tag{43}$$

☞ Advantage of Shapley Value

Theorem 5.5

Shapley value is the only imputation, satisfying

- Efficiency Axiom (有效性公理);
- Symmetry Axiom (对称公理);
- Additivity Axiom (可加性公理).

谢政,《对策论导引》, 科学出版社,北京,2010.

☞ A Formula for Calculating Shapley Value

- Step 1: Construct a sequence of vectors ℓ_k:

$$\begin{cases} \ell_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^2; \\ \ell_{k+1} = \begin{bmatrix} \ell_k + \mathbf{1}_{2^k} \\ \ell_k \end{bmatrix} \in \mathbb{R}^{2^{k+1}}, \\ \qquad k = 1, 2, 3, \cdots . \end{cases} \tag{44}$$

Example 5.6

$$\ell_2 = \begin{bmatrix} \ell_1 + \mathbf{1}_2 \\ \ell_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

- Step 2: Construct $\eta_k \in \mathbb{R}^{2^k}$:

$$\eta_k = (\ell_k)!(k\mathbf{1}_{2^k} - \ell_k)!. \tag{45}$$

Example 5.7

$$\eta_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \cdots .$$

- Step 3: Set $\zeta := \eta_{n-1}$.
  Split ζ equally into k blocks:

$$\zeta = \begin{bmatrix} \zeta_k^1 \\ \zeta_k^2 \\ \vdots \\ \zeta_k^k \end{bmatrix}, \quad k = 1, 2, 2^2, \cdots, 2^{n-1}.$$

- Step 4: Define Ξ_n as:

$$\Xi_n = \frac{1}{n!}\left[\begin{pmatrix}\zeta_1 \\ -\zeta_1\end{pmatrix} \quad \begin{pmatrix}\zeta_2^1 \\ -\zeta_2^1 \\ \zeta_2^2 \\ -\zeta_2^2\end{pmatrix} \quad \begin{pmatrix}\zeta_4^1 \\ -\zeta_4^1 \\ \zeta_4^2 \\ -\zeta_4^2 \\ \zeta_4^3 \\ -\zeta_4^3 \\ \zeta_4^4 \\ -\zeta_4^4\end{pmatrix} \quad \cdots \quad \begin{pmatrix}\zeta_{2^{n-1}}^1 \\ -\zeta_{2^{n-1}}^1 \\ \zeta_{2^{n-1}}^2 \\ -\zeta_{2^{n-1}}^2 \\ \vdots \\ \zeta_{2^{n-1}}^{2^{n-1}} \\ -\zeta_{2^{n-1}}^{2^{n-1}}\end{pmatrix}\right]. \tag{46}$$

Theorem 5.8

$$\varphi(v) = V_v \Xi_n. \tag{47}$$

☞ References:

- Y. Wang, D. Cheng, X. Liu, Matrix expression of Shapley values and its application to distributed resource allocation, *Sci. China Inform. Sci.*, Vol. 62, 022201:1-022201:11, 1019.
- H. Li, S. Wang, A. Liu, M. Xia, Simplification of Shapley value for cooperative games via minimum carrier, *Contr. Theor. Tech.*, Vol. 19, 157-169, 2021.
- X. Xia, H. Li, X. Ding, Y. Liu, Matrix approach to calculation of Banzhaf value with applications, *Contr. Theor. Appl.*, Vol. 37, No. 2, 446-452, 2020.

Example 5.9

We calculate some Ξ_n for small n.

- $n = 2$:

$$\ell_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T;$$

$$\eta_1 = \begin{bmatrix} 1!(2-1-1)! & 0!(2-1-0)! \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\Xi_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Example 5.9(cont's)

- $n = 3$:

$$\Xi_3 = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \\ 2 & -1 & -1 \\ -2 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \\ -2 & -2 & -2 \end{bmatrix}.$$

Example 5.9(cont's)

- $n = 4$:

$$\Xi_4 = \frac{1}{24}\begin{bmatrix} 6 & 6 & 6 & 6 \\ 2 & 2 & 2 & -6 \\ 2 & 2 & -6 & 2 \\ 6 & 6 & -2 & -2 \\ 2 & -6 & 2 & 2 \\ 6 & -2 & 6 & -2 \\ 6 & -2 & -2 & 6 \\ 6 & -6 & -6 & -6 \\ -6 & 2 & 2 & 2 \\ -2 & 6 & 6 & -2 \\ -2 & 6 & -2 & 6 \\ -6 & 6 & -6 & -6 \\ -2 & -2 & 6 & 6 \\ -6 & -6 & 6 & -6 \\ -6 & -6 & -6 & 6 \\ -6 & -6 & -6 & -6 \end{bmatrix}.$$

Example 5.10

Recall Example 1.3 (selling horse).

$$V_v = \begin{bmatrix} 110 & 100 & 110 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using formula (47), The Shapley value is

$$\varphi(v) = V_v \Xi_3 = \begin{bmatrix} 71.67 & 16.67 & 21.67 \end{bmatrix}.$$

VI. Final Remarks

☞ Remarks on Cooperative Game

(i) Cooperative game ($G = (N, v)$) is another kind of games (vs non-cooperative game).

(ii) Constant sum game has a natural cooperative game structure.

(iii) Unanimity games form a basis for cooperative games ($\sim \mathbb{R}^{2^n-1}$).

(iv) Normal games are canonical form of cooperative games.

(v) Imputation is the purpose of cooperative game theory. Shapley value is one of the useful imputations.

谢　谢　！

Q&A