Potential Game Theory Based on STP

Lesson Two (第二讲)

Lecturer: Jiandong Zhu (朱建栋)

(School of Mathematical Sciences, Nanjing Normal University, Nanjing)
南京师范大学数学科学学院

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Outline

Modern game theory

Modern game theory began with the idea of mixed-strategy equilibria in two-person zero-sum games and its proof by John von Neumann. His paper was followed by the 1944 book Theory of Games and Economic Behavior, co-written with Oskar Morgenstern.

Introduction, Definition of Game

Definition 1. [1] A **finite game** is a triple $\mathcal{G} = (\mathcal{N},\ \mathcal{S},\ \mathcal{C})$, where

(i) $\mathcal{N} = \{1, 2, \cdots, n\}$ is the set of players;

(ii) $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$, where each $\mathcal{S}_i = \{s^i_1, s^i_2, \cdots, s^i_{k_i}\}$ is the strategy set of player i;

(iii) $\mathcal{C} = \{c_1, c_2, \cdots, c_n\}$ is the set of payoff functions, where every $c_i : \mathcal{S} \to \mathbb{R}$ is the payoff function of player i.

The finite game defined above is called a **normal form game** in [2].

[1] Monderer D, Shapley LS (1996) Potential games. Games and Economic Behavior, 14, 124-143.

[2] Sandholm WH, (2010) Decompositions and potentials for normal form games. Games and Economic Behavior, 70, 446-456.

Introduction, Definition of Game

Let $c^{\mu}_{i_1 i_2 \cdots i_n} = c_\mu(s^1_{i_1}, s^2_{i_2}, \cdots, s^n_{i_n})$ where $1 \le i_s \le k_s$ and $s = 1, 2, \cdots, n$. Then the finite game can be described by the arrays

$$C_\mu = \{c^{\mu}_{i_1 i_2 \cdots i_n} | \ 1 \le i_s \le k_s, \ s = 1, 2, \cdots, n\} \tag{1}$$

with $\mu = 1, 2, \cdots, n$.

Given n and $k_1, \ldots, k_n$, the set of all finite games is a linear space with dimension $d = nk_1k_2 \cdots k_n$.

Particularly, for a 2-player game, the $k_1 \times k_2$ matrices $C_1 = (c^1_{ij})$ and $C_2 = (c^2_{ij})$ are payoffs of players 1 and 2 respectively. Therefore, a 2-player finite game is also called a bi-matrix game, which is usually denoted by the simple notation $\mathcal{G} = (C_1, C_2)$.

Introduction, Examples of Game

The game of ’rock, paper, scissors’ with 2 players:

		B		
		r	p	s
A	r	$(0, 0)$	$(-1, 1)$	$(1, -1)$
	p	$(1, -1)$	$(0, 0)$	$(-1, 1)$
	s	$(-1, 1)$	$(1, -1)$	$(0, 0)$

$$C_1 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Question

How can we describe a 3-player game in a matrix form?

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Introduction, Examples of Game

The game of 'palm up, palm down' with 3 players:

A B C	uuu	uud	udu	udd	duu	dud	ddu	ddd
c_1	0	1	1	-2	-2	1	1	0
c_2	0	1	-2	1	1	-2	1	0
c_3	0	-2	1	1	1	1	-2	0

Payoff matrix is defined as

$$P = \begin{bmatrix} 0 & 1 & 1 & -2 & -2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 1 & 1 & 1 & -2 & 0 \end{bmatrix}.$$

This description of finite games was proposed in [3].

[3] D. Cheng, On finite potential games, Automatica, 50, 1793-1801, 2014.

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The matrix form of payoff function:

$$c_1(x_1, x_2, x_3) = [0\ 1\ 1\ -2\ -2\ 1\ 1\ 0]x_1x_2x_3,$$

where

$$x_i \in \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

matrix forms of payoff matrices

In general, each payoff function can be rewritten in the matrix form based on STP as follows:

$$c_i(x_1, x_2, \cdots, x_n) = V_i^c x_1 x_2 \cdots x_n,$$

where $x_j \in \Delta_{k_j}$, $i, j = 1, 2, \ldots, n$.
The payoff matrix is an $n \times k_1 k_2 \cdots k_n$ matrix:

$$P = \begin{bmatrix} V_1^c \\ V_2^c \\ \vdots \\ V_n^c \end{bmatrix}.$$

Obviously, the **dimension** of the linear space composed of all $n \times k_1 k_2 \cdots k_n$ matrices is $n k_1 k_2 \cdots k_n$.

Introduction, Nash Equilibria

A strategy profile $s = (s_1,\ s_2,\ \cdots,\ s_n) \in S$ is a **Nash equilibrium (NE)** if

$$f_i(s_i, s^{-i}) \geq f_i(x_i, s^{-i}) \quad \forall i,\ x_i \in S_i.$$

Example: Prisoner's Dilemma

		B	
		s	b
A	s	$(-1, -1)$	$(-10, 0)$
	b	$(0, -10)$	$(-8, -8)$

s=silent; b=betray. Nash Equilibrium: (b,b).

$$C_1 = \begin{bmatrix} -1 & -10 \\ 0 & -8 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1 & 0 \\ -10 & -8 \end{bmatrix}$$

Introduction, Nash Equilibria

Nash's Existence Theorem

If we allow mixed strategies, then every game with a finite number of players in which each player can choose from finitely many pure strategies has at least one Nash equilibrium.

Introduction, Nash Equilibria

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		B	
		head	tail
A	head	$(3, -3)$	$(-2, 2)$
	tail	$(-2, 2)$	$(1, -1)$

If we are allowed to take a mixed strategy, we can take $\frac{1}{3}$ head and $\frac{2}{3}$ tail.

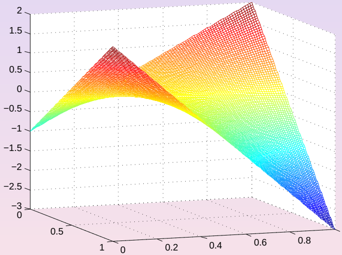
Let $P(A = head) = x$ and $P(B = head) = y$. Then we have

$$c_1 = 3 \cdot xy + 1 \cdot (1-x)(1-y) - 2 \cdot (1-x)y - 2 \cdot x(1-y) = 8xy - 3x - 3y + 1;$$

The payoff function of A is $c_1(x, y) = 8xy - 3x - 3y + 1$.

The Nash equilibrium is $(x^*, y^*) = (\frac{3}{8},\ \frac{3}{8})$ and

$$c_1(x, \frac{3}{8}) = 8x\frac{3}{8} - 3x - 3\frac{3}{8} + 1 = -\frac{1}{8}, \quad c_2(x, y^*) = \frac{1}{8}, \quad \forall\, x.$$



$$c_1(x, y) = 8xy - 3x - 3y + 1.$$

Definition of Potential Game

Question

What kind of games have a Nash Equilibrium under pure strategies?

Definition.

(Monderer & Shapley, 1996) A finite game $\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$ is said to be potential if there exists a function $p : \mathcal{S} \to \mathbb{R}$, called the potential function, such that

$$c_i(x, s^{-i}) - c_i(y, s^{-i}) = p(x, s^{-i}) - p(y, , s^{-i})$$

for all $x, y \in \mathcal{S}_i$, $s^{-i} \in \mathcal{S}^{-i}$ $i = 1, 2, \cdots, n$, where $\mathcal{S}^{-i} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_n$.

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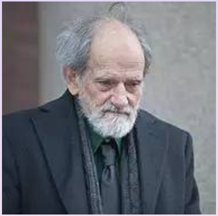
potential games

Theorem

(Monderer & Shapley, 1996) Every finite potential game possesses a pure Nash equilibrium.



Question

(Monderer & Shapley, 1996) How can we test whether a finite game is potential?

Conservative vector field

In vector calculus, a **conservative vector field (potential field)** is a gradient field of a scalar function called a **potential function**.

A vector field is a **conservative field** if and only if the line integral is **path independent**.

Gradient Theorem

A conservative vector field $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), \ldots, c_n(\mathbf{x}))^{\mathrm{T}}$ satisfies

$$\int_C \mathbf{c}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = p(B) - p(A),$$

where C is any path from point A to point B, and $p(\cdot)$ is the **potential function**.

Conservative vector field

Gradient Theorem

A conservative vector field $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), c_2(\mathbf{x}))^{\mathrm{T}}$ satisfies

$$\int_C c_1(\mathbf{x})\mathrm{d}x_1 + c_2(\mathbf{x})\mathrm{d}x_2 = p(B) - p(A),$$

where C is a path form point A to point B, and $p(\cdot)$ is the **potential function**.

The potential function $p(\mathrm{x})$ is

$$p(x_1, x_2) = \int_{(a,b)}^{(x_1,x_2)} c_1(\mathbf{x})\mathrm{d}x_1 + c_2(\mathbf{x})\mathrm{d}x_2.$$

Let p_1 and p_2 be potentials for a conservative vector field. Then there exists a constant c such that

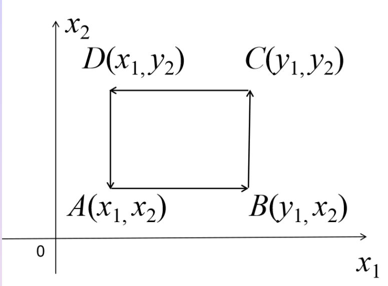
$$p_1(x_1, x_2) - p_2(x_1, x_2) = c \text{ for every } (x_1, x_2).$$

Conservative vector field

Vector field $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), c_2(\mathbf{x}))^{\mathrm{T}}$ is a conservative field if and only if

$$\oint c_1(\mathbf{x})\mathrm{d}x_1 + c_2(\mathbf{x})\mathrm{d}x_2 = 0$$

for every closed-loop.



In particular, consider the above closed loop. If vector field $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), c_2(\mathbf{x}))^{\mathrm{T}}$ is conservative, then

$$\int_{x_1}^{y_1} c_1(\mathbf{x})\mathrm{d}x_1 + \int_{x_2}^{y_2} c_2(\mathbf{x})\mathrm{d}x_2 + \int_{y_1}^{x_1} c_1(\mathbf{x})\mathrm{d}x_1 + \int_{y_2}^{x_2} c_2(\mathbf{x})\mathrm{d}x_2 = 0.$$

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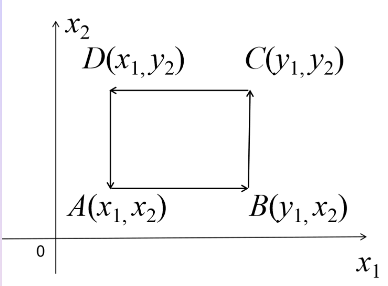

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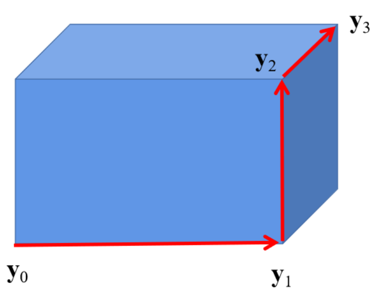
potential games

Definition

(Monderer & Shapley, 1996) A **path** in $\mathcal{S}$ is a sequence $\gamma = (\mathbf{y}_0, \mathbf{y}_1, \ldots)$ such that for every $k \geq 1$ there exists a unique player, say Player i, such that $\mathbf{y}_k = (y_{k-1}^{-i}, x)$ for some $x \neq y_{k-1}^i$ in $\mathcal{S}$.



potential games

Definition (Monderer & Shapley, 1996)

For a finite path $\gamma = (\mathbf{y}_0, \mathbf{y}_1, \ldots \mathbf{y}_N)$ and for a vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ of payoff functions $c_i(\mathbf{x})$, the **total payoff** along γ is defined as

$$I(\gamma, \mathbf{c}) = \sum_{k=1}^{N} [c_{i_k}(\mathbf{y}_k) - c_{i_k}(\mathbf{y}_{k-1})],$$

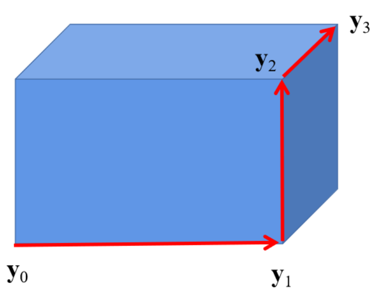
where i_k is the unique deviator at step k.

The total payoff is similar to the line integral along γ:

$$\int_\gamma \mathbf{c}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \sum_{k=1}^{N} \int_{\mathbf{y}_{k-1}}^{\mathbf{y}_k} c_{i_k}(\mathbf{x}) \mathrm{d}x_{i_k}.$$

potential games

For the finite path $\gamma = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$



the total payoff is

$$I(\gamma, \mathbf{c}) = [c_1(\mathbf{y}_1) - c_1(\mathbf{y}_0)] + [c_3(\mathbf{y}_2) - c_3(\mathbf{y}_1)] + [c_2(\mathbf{y}_3) - c_2(\mathbf{y}_2)],$$

which is similar to the line integral

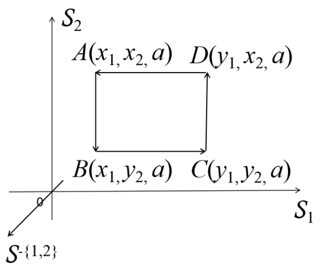
$$\int_\gamma \mathbf{c}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int_{\mathbf{y}_0}^{\mathbf{y}_1} c_1(\mathbf{x}) \mathrm{d}x_1 + \int_{\mathbf{y}_1}^{\mathbf{y}_2} c_3(\mathbf{x}) \mathrm{d}x_3 + \int_{\mathbf{y}_2}^{\mathbf{y}_3} c_2(\mathbf{x}) \mathrm{d}x_2.$$

potential games

Theorem (Monderer & Shapley, 1996)

Let $\mathcal{G}$ be a finite game with payoff vector $\mathbf{c}$. The following claims are equivalent:
(1) $\mathcal{G}$ is a potential game;
(2) $I(\gamma, \mathbf{c}) = 0$ for every finite closed path γ;
(3) $I(\gamma, \mathbf{c}) = 0$ for every simple closed path γ of length 4.

path4.pdf path4.pdf



$$[c_2(B)-c_2(A)]+[c_1(C)-c_1(B)]+[c_2(D)-c_2(C)]+[c_1(A)-c_1(D)]=0.$$

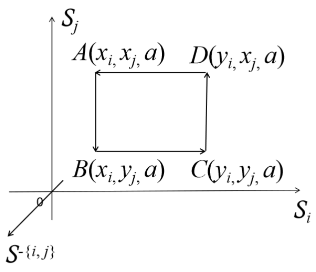
potential games

Theorem (Monderer & Shapley, 1996)

$\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$ is a potential game iff for every $i, j \in \mathcal{N}$, for every $a \in \mathcal{S}^{-\{i,j\}}$, and for every $x_i, y_i \in \mathcal{S}_i$ and $x_j, y_j \in \mathcal{S}_j$,

$$[c_j(B) - c_j(A)] + [c_i(C) - c_i(B)] + [c_j(D) - c_j(C)] + [c_i(A) - c_i(D)] = 0.$$

It is called a **four-cycle equation** in (Sandholm 2010).



Question: How many equations are needed to check?

potential games

Question

How many equations are needed to check for a finite game with n players and k strategies for each player?

By (Monderer & Shapley, 1996), the number of equations corresponding to simple closed loops with length 4 is

$$C_n^2 k^{n-2} C_k^2 C_k^2 = \frac{n(n-1)k^n(k-1)^2}{6} = O(n^2 k^{n+2}).$$

The theoretical minimum value of the number of equations is

$$nk^n - (k^n + nk^{n-1} - 1) = (n-1)k^n - nk^{n-1} + 1 = O(nk^n).$$

potential games

U is a potential game if and only if there is a potential function V and auxiliary functions $W_p : \mathcal{S}^{-p} \to \mathbf{R}$ such that

$$U_p(s) = V(s) + W_p(s^{-p}) \;\; \forall\, s \in \mathcal{S}, \forall\, p \in \mathcal{N}. \tag{2}$$

Proof. ($\Leftarrow$) If (2) holds, then

$$U_p(x, s^{-p}) = V(x, s^{-p}) + W_p(s^{-p}) \tag{3}$$

and

$$U_p(y, s^{-p}) = V(y, s^{-p}) + W_p(s^{-p}). \tag{4}$$

From (3)–(4), it follows that

$$U_p(x, s^{-p}) - U_p(y, s^{-p}) = V(x, s^{-p}) - V(y, s^{-p}).$$

Therefore, U is a potential game with the potential function $V(x)$.

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Proof. $(\Rightarrow)$ Assume that U is a potential game with the potential function $V(x)$, i.e.,

$$U_p(x, s^{-p}) - U_p(y, s^{-p}) = V(x, s^{-p}) - V(y, s^{-p})$$

for any $x, y \in \mathcal{S}_i$ and $s^{-p} \in \mathcal{S}^{-p}$. So,

$$U_p(x, s^{-p}) - V(x, s^{-p}) = U_p(y, s^{-p}) - V(y, s^{-p}). \tag{6}$$

Let $W_p(s) = U_p(s) - V(s)$. Then (6) implies that $W_p(s)$ is independent of s_p, which is rewritten as $W_p(s^{-p})$. Therefore,

$$U_p(s) = V(s) + W_p(s^{-p}). \tag{7}$$

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Let $W_p(s) = U_p(s) - V(s)$. Then (6) implies that $W_p(s)$ is independent of s_p, which is rewritten as $W_p(s^{-p})$. Therefore,

$$U_p(s) = V(s) + W_p(s^{-p}). \tag{7}$$

potential games

U is a potential game if and only if there is a potential function V and auxiliary functions $W_p : \mathcal{S}^{-p} \to \mathbf{R}$ such that

$$U_p(s) = V(s) + W_p(s^{-p}) \;\; \forall\, s \in \mathcal{S}, \forall\, p \in \mathcal{N}. \tag{5}$$

Proof. $(\Rightarrow)$ Assume that U is a potential game with the potential function $V(x)$, i.e.,

$$U_p(x, s^{-p}) - U_p(y, s^{-p}) = V(x, s^{-p}) - V(y, s^{-p})$$

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By using the matrix form based on STP, U is a potential game iff its payoff matrix U has the form

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_p \end{bmatrix} = \begin{bmatrix} V \\ V \\ \vdots \\ V \end{bmatrix} + \begin{bmatrix} ? \\ ? \\ \vdots \\ ? \end{bmatrix}.$$

Let $x \in \Delta_{n_1}$, $y \in \Delta_{n_2}$ and $z \in \Delta_{n_3}$. Then

$$xz = (I_{n_1} \otimes \mathbf{1}_{n_2}^{\mathrm{T}} \otimes I_{n_3})xyz.$$

Proof. $(I_{n_1} \otimes \mathbf{1}_{n_2}^{\mathrm{T}} \otimes I_{n_3})(x \otimes y \otimes z) = x \otimes 1 \otimes z = xz.$

potential games

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potential games

U is a potential game iff its payoff matrix U has the form

$$U = \begin{bmatrix} V \\ V \\ \vdots \\ V \end{bmatrix} + \begin{bmatrix} W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ W_2(I_k \otimes \mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-2}}) \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ W_n(I_{k^{n-1}} \otimes \mathbf{1}_k^{\mathrm{T}}) \end{bmatrix}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of a n-dimensional linear space. Then

$$\dim(\mathcal{X} + \mathcal{Y}) = \dim(\mathcal{X}) + \dim(\mathcal{Y}) - \dim(\mathcal{X} \cap \mathcal{Y}).$$

So the dimension of the linear space composed of potential games is $k^n + nk^{n-1} - 1$. (Sandholm, Games Econ Behav, 2010; Monderer D, Shapley, Games Econ Behav, 1996)

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potential games

Theorem (Hino, Int J Game Theory 2011)

$\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$ is a potential game iff for every $i, j \in \mathcal{N}$, for every $a \in \mathcal{S}^{-\{i,j\}}$, and for every $x_i \in \mathcal{S}_i$ and $x_j \in \mathcal{S}_j$,

$$[c_j(B)-c_j(A)]+[c_i(C)-c_i(B)]+[c_j(D)-c_j(C)]+[c_i(A)-c_i(D)]=0,$$

where $A = (x_i, x_j, a)$, $B = (x_i + 1, x_j, a)$, $C = (x_i + 1, x_j + 1, a)$, and $D = (x_i, x_j + 1, a)$. The number of four-cycle equations is

$$C_n^2 k^{n-2} C_k^2 C_k^2 = O(n^2 k^{n+2}).$$

By (Hino, 2011), the number of equations is

$$C_n^2 k^{n-2} (k-1)^2 = O(n^2 k^n).$$

The minimum value is $(n-1)k^n - nk^{n-1} + 1 = O(nk^n)$.

[4] Y. Hino, An improved algorithm for detecting potential games, Int J Game Theory (2011) 40:199-205.

Potential equation

A finite game U is a potential game iff there are exists row vectors V and W_i such that

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} = \begin{bmatrix} V \\ V \\ \vdots \\ V \end{bmatrix} + \begin{bmatrix} W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ W_2(I_k \otimes \mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-2}}) \\ \vdots \\ W_n(I_{k^{n-1}} \otimes \mathbf{1}_k^{\mathrm{T}}) \end{bmatrix}.$$

or

$$\begin{bmatrix} U_1 \\ U_2 - U_1 \\ \vdots \\ U_n - U_1 \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ W_2(I_k \otimes \mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-2}}) - W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ \vdots \\ W_n(I_{k^{n-1}} \otimes \mathbf{1}_k^{\mathrm{T}}) - W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \end{bmatrix}.$$

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or

$$\begin{bmatrix} -\mathbf{1}_k \otimes I_{k^{n-1}} & I_k \otimes \mathbf{1}_k \otimes I_{k^{n-2}} & & & \\ -\mathbf{1}_k \otimes I_{k^{n-1}} & & I_{k^2} \otimes \mathbf{1}_k \otimes I_{k^{n-3}} & & \\ \vdots & & & \ddots & \\ -\mathbf{1}_k \otimes I_{k^{n-1}} & & & & I_{k^{n-1}} \otimes \mathbf{1}_k \end{bmatrix} \xi = \begin{bmatrix} (U_2 - U_1)^{\mathrm{T}} \\ \vdots \\ (U_n - U_1)^{\mathrm{T}} \end{bmatrix}.$$

The potential equation is dented by $\Psi\xi = b$, where Ψ is an $(n-1)k^n \times nk^{n-1}$ matrix.

Potential equation

Theorem (Cheng, Automatica, 2014)

The game $\mathcal{G} = (\mathcal{N},\ \mathcal{S},\ \mathcal{C})$ is potential if and only if the potential equation
$\Psi\xi = b$ has a solution ξ.

Lemma (Cheng, Automatica, 2014)

$$\Psi \mathbf{1}_{nk^{n-1}} = 0; \qquad \text{rank}\Psi = nk^{n-1} - 1.$$

We only need to prove that the dimension of $\Psi\xi = 0$ is 1. Assume that $\Psi\xi = 0$, prove that $\xi = a\mathbf{1}_{nk^{n-1}}$ for some a.

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Potential equation

Question

What is the relationship between the four-cycle equation and the potential equation?

For the case of $n = 2$, the potential equation is

$$\left[-\mathbf{1}_{k_1} \otimes I_{k_2} \quad I_{k_1} \otimes \mathbf{1}_{k_2}\right] \xi = b.$$

Theorem

The bi-matrix game $\mathcal{G} = (C_1,\ C_2)$ is a potential game if and only if

$$B_{k_1}(C_2 - C_1)B_{k_2}^{\mathrm{T}} = 0, \qquad (8)$$

where $B_k = [I_{k-1},\ -\mathbf{1}_{k-1}]$

[5] Xinyun Liu, Jiandong Zhu, On potential equations of finite games, Automatica, 68, 245-253, 2016.

Proof. Let $D_k := [I_{k-1},\ 0] \in \mathbb{R}^{(k-1)\times k}$. Then it is easy to see that

$$B_k D_k^{\mathrm{T}} = I_{k-1}, \quad D_k \delta_k^k = B_k \mathbf{1}_k = 0. \qquad (9)$$

Construct two matrices

$$E = [-\delta_{k_1}^{k_1} \otimes I_{k_2},\ B_{k_1}^{\mathrm{T}} \otimes \delta_{k_2}^{k_2},\ B_{k_1}^{\mathrm{T}} \otimes B_{k_2}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{k_1k_2 \times k_1k_2}$$

$$F = [-\mathbf{1}_{k_1} \otimes I_{k_2},\ D_{k_1}^{\mathrm{T}} \otimes \mathbf{1}_{k_2},\ D_{k_1}^{\mathrm{T}} \otimes D_{k_2}^{\mathrm{T}}] \in \mathbb{R}^{k_1k_2 \times k_1k_2}.$$

Then a straightforward calculation shows that

$$\begin{array}{lcl} EF & = & \begin{bmatrix} -(\delta_{k_1}^{k_1})^{\mathrm{T}} \otimes I_{k_2} \\ B_{k_1} \otimes (\delta_{k_2}^{k_2})^{\mathrm{T}} \\ B_{k_1} \otimes B_{k_2} \end{bmatrix} [-\mathbf{1}_{k_1} \otimes I_{k_2},\ D_{k_1}^{\mathrm{T}} \otimes \mathbf{1}_{k_2},\ D_{k_1}^{\mathrm{T}} \otimes D_{k_2}^{\mathrm{T}}] \\ & = & \begin{bmatrix} I_{k_2} & 0 & 0 \\ 0 & I_{k_1-1} & 0 \\ 0 & 0 & I_{(k_1-1)(k_2-1)} \end{bmatrix} = I_{k_1k_2} \end{array} \quad lcl$$

Potential equation

So the potential equation is equivalent to $E\Psi\xi = Eb$. It is easy to check that

$$\begin{aligned}
E[\Psi, \;\; b] &= \begin{bmatrix} -(\delta_{k_1}^{k_1})^{\mathrm{T}} \otimes I_{k_2} \\ B_{k_1} \otimes (\delta_{k_2}^{k_2})^{\mathrm{T}} \\ B_{k_1} \otimes B_{k_2} \end{bmatrix} [-\mathbf{1}_{k_1} \otimes I_{k_2}, \quad I_{k_1} \otimes \mathbf{1}_{k_2}, \;\; b] \\
&= \begin{bmatrix} I_{k_2} & -(\delta_{k_1}^{k_1})^{\mathrm{T}} \otimes \mathbf{1}_{k_2} & -((\delta_{k_1}^{k_1})^{\mathrm{T}} \otimes I_{k_2})b \\ 0 & B_{k_1} & (B_{k_1} \otimes (\delta_{k_2}^{k_2})^{\mathrm{T}})b \\ 0 & 0 & (B_{k_1} \otimes B_{k_2})b \end{bmatrix} \\
&= \begin{bmatrix} I_{k_2} & 0 & -\mathbf{1}_{k_2} & -((\delta_{k_1}^{k_1})^{\mathrm{T}} \otimes I_{k_2})b \\ 0 & I_{k_1-1} & -\mathbf{1}_{k_1-1} & (B_{k_1} \otimes (\delta_{k_2}^{k_2})^{\mathrm{T}})b \\ 0 & 0 & 0 & (B_{k_1} \otimes B_{k_2})b \end{bmatrix}. \qquad (10)
\end{aligned}$$

So the potential equation is solvable if and only if

$$(B_{k_1} \otimes B_{k_2})b = 0, \text{ i.e., } B_{k_1}(C_2 - C_1)B_{k_2}^{\mathrm{T}} = 0. \qquad (11)$$

Potential equation

Corollary

The bi-matrix game $\mathcal{G} = (C_1,\ C_2)$ is a potential game if and only if

$$r_{ij} - r_{ik_2} - r_{k_1 j} + r_{k_1 k_2} = 0 \tag{12}$$

for every $i = 1, 2, \cdots, k_1 - 1$ and $j = 1, 2, \cdots, k_2 - 1$, where $(r_{ij}) = C_2 - C_1$.

$$\begin{aligned}
& r_{ij} - r_{ik_2} - r_{k_1 j} + r_{k_1 k_2} \\
= \ & c_2(i,j) - c_1(i,j) - c_2(i,k_2) + c_1(i,k_2) \\
& -c_2(k_1,j) + c_1(k_1,j) + c_2(k_1,k_2) - c_1(k_1,k_2) \\
= \ & [c_1(k_1,j) - c_1(i,j)] + [c_2(k_1,k_2) - c_2(k_1,j)] \\
& +[c_1(i,k_2) - c_1(k_1,k_2)] + [c_2(i,j) - c_2(i,k_2)]
\end{aligned} \tag{13}$$

So the condition in the theorem is just a set of four-cycle equations.

potential games

Given the strategy set for bi-matrix games, the set of all the relative payoff matrices of potential bi-matrix games is a $(k_1 + k_2 - 1)$-dimensional subspace, which is isomorphic to

$$\mathcal{P} = \{b \in \mathbb{R}^{k_1 k_2} | \ (B_{k_1} \otimes B_{k_2})b = 0\}. \quad (14)$$

Lemma

Consider a linear subspace of $\mathbb{R}^n$ as follows:

$$\mathcal{X} = \{v \in \mathbb{R}^n | \, Bv = 0\}. \quad (15)$$

If B has a **full row rank**, then the orthogonal projection of u onto $\mathcal{X}$ is

$$\mathrm{Proj}_{\mathcal{X}} u = (I_n - B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}B)u. \quad (16)$$

potential games

Now we consider the orthogonal projection onto the potential subspace.

Theorem

Consider a bi-matrix game $\mathcal{G} = (C_1,\ C_2)$, where $C_1, C_2 \in \mathbb{R}^{k_1\times k_2}$. Denote the relative payoff matrix by $R=(r_{ij})=C_2-C_1$ and let $H_k=I_k-\frac{1}{k}\mathbf{1}_k\mathbf{1}_k^{\mathrm{T}}$. Then

$$\mathrm{Proj}_{\mathcal{P}}\mathrm{V_r}(R) = (I_{k_1k_2} - H_{k_1} \otimes H_{k_2})\mathrm{V_r}(R). \tag{17}$$

Proof. Let $\tilde{B} = B_{k_1} \otimes B_{k_2}$. By Lemma, we have

$$\begin{aligned} & \mathrm{Proj}_{\mathcal{P}}\mathrm{V_r}(R) \\ = & \ (I_{k_1k_2} - \tilde{B}^{\mathrm{T}}(\tilde{B}\tilde{B}^{\mathrm{T}})^{-1}\tilde{B})\mathrm{V_r}(R) \\ = & \ (I_{k_1k_2} - B_{k_1}^{\mathrm{T}}(B_{k_1}B_{k_1}^{\mathrm{T}})^{-1}B_{k_1} \otimes B_{k_2}^{\mathrm{T}}(B_{k_2}B_{k_2}^{\mathrm{T}})^{-1}B_{k_2})\mathrm{V_r}(R). \end{aligned}$$

potential games

A straightforward computation shows that

$$
\begin{aligned}
& B_k^{\mathrm{T}}(B_k B_k^{\mathrm{T}})^{-1} B_k \\
= \; & \begin{bmatrix} I_{k-1} \\ -\mathbf{1}_{k-1}^{\mathrm{T}} \end{bmatrix} (I_{k-1} + \mathbf{1}_{k-1}\mathbf{1}_{k-1}^{\mathrm{T}})^{-1} [I_{k-1} \quad -\mathbf{1}_{k-1}] \\
= \; & \begin{bmatrix} I_{k-1} \\ -\mathbf{1}_{k-1}^{\mathrm{T}} \end{bmatrix} (I_{k-1} - \frac{1}{k}\mathbf{1}_{k-1}\mathbf{1}_{k-1}^{\mathrm{T}}) [I_{k-1} \quad -\mathbf{1}_{k-1}] \\
= \; & \begin{bmatrix} I_{k-1} - \frac{1}{k}\mathbf{1}_{k-1}\mathbf{1}_{k-1}^{\mathrm{T}} & -\frac{1}{k}\mathbf{1}_{k-1} \\ -\frac{1}{k}\mathbf{1}_{k-1}^{\mathrm{T}} & \frac{k-1}{k} \end{bmatrix} \\
= \; & I_k - \frac{1}{k}\mathbf{1}_k\mathbf{1}_k^{\mathrm{T}} = H_k.
\end{aligned} \tag{18}
$$

It follows that (17) holds. □

potential games

Theorem

Consider a bi-matrix game $\mathcal{G} = (C_1, C_2)$, where $C_1, C_2 \in \mathbb{R}^{k_1 \times k_2}$. Let $R = (r_{ij}) = C_2 - C_1$. Then the following statements are equivalent:

(i) $\mathcal{G}$ is a potential game;

(ii) $H_{k_1} R H_{k_2} = 0$, where $H_k = I_k - \frac{1}{k}\mathbf{1}_k \mathbf{1}_k^{\mathrm{T}}$;

(iii) $r_{ij} = r_{i-\text{ave}} + r^{j-\text{ave}} - r_{\text{ave}}$ for all $i = 1, 2, \cdots, k_1$ and $j = 1, 2, \cdots, k_2$, where

$$r_{i-\text{ave}} = \frac{1}{k_2}\sum_{\mu=1}^{k_2} r_{i\mu}, \quad r^{j-\text{ave}} = \frac{1}{k_1}\sum_{\lambda=1}^{k_1} r_{\lambda j}, \tag{19}$$

$$r_{\text{ave}} = \frac{1}{k_1 k_2}\sum_{\lambda=1}^{k_1}\sum_{\mu=1}^{k_2} r_{\lambda\mu}. \tag{20}$$

potential games

Proof. Obviously, $\mathcal{G}$ is a potential game if and only if $\mathrm{Proj}_{\mathcal{P}}\mathrm{V}_{\mathrm{r}}(R) = \mathrm{V}_{\mathrm{r}}(R)$, where $\mathcal{P}$ is the potential subspace. Therefore, we have that $\mathcal{G}$ is potential if and only if

$$(H_{k_1} \otimes H_{k_2})\mathrm{V}_{\mathrm{r}}(R) = 0, \text{ i.e. } H_k R H_k = 0.$$

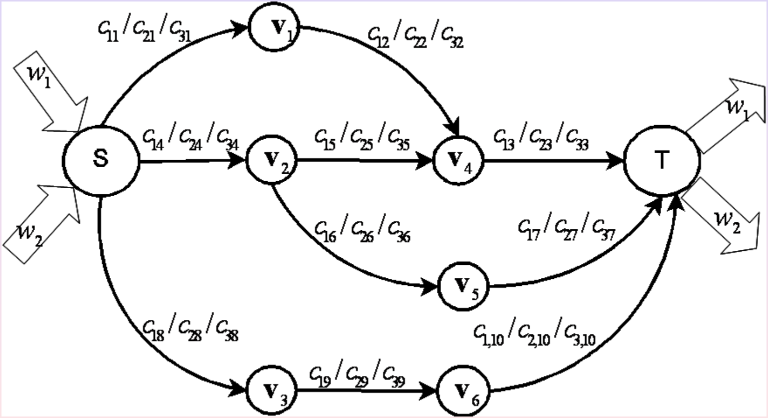
Moreover, a straightforward calculation shows that

$$\begin{aligned} & H_k R H_k \\ = & (I_{k_1} - \frac{1}{k_1}\mathbf{1}_{k_1}\mathbf{1}_{k_1}^{\mathrm{T}})R(I_{k_2} - \frac{1}{k_2}\mathbf{1}_{k_2}\mathbf{1}_{k_2}^{\mathrm{T}}) \\ = & R - \frac{1}{k_1}\mathbf{1}_{k_1}\mathbf{1}_{k_1}^{\mathrm{T}}R - \frac{1}{k_2}R\mathbf{1}_{k_2}\mathbf{1}_{k_2}^{\mathrm{T}} + \frac{\mathbf{1}_{k_1}^{\mathrm{T}}R\mathbf{1}_{k_2}}{k_1 k_2}\mathbf{1}_{k_1}\mathbf{1}_{k_2}^{\mathrm{T}}. \end{aligned} \quad (21)$$

From (19)-(21), the equivalence between (ii) and (iii) follows. □

weighted network congestion games

Consider an example of weighted network congestion games (WNCG) addressed in Lemma 1 of Fotakis, Kontogiannis, and Spirakis (2005).



A weighted congestion game

With simple calculations, we get the relative payoff matrix $R = w_2 P_2 - w_1 P_1$, where P_1 and P_2 are given as follows:

$$P_1 = \begin{bmatrix} c_{31}+c_{32}+c_{33} & c_{11}+c_{12}+c_{33} & c_{11}+c_{12}+c_{13} & c_{11}+c_{12}+c_{13} \\ c_{33}+c_{14}+c_{15} & c_{33}+c_{34}+c_{35} & c_{13}+c_{34}+c_{15} & c_{13}+c_{14}+c_{15} \\ c_{14}+c_{16}+c_{17} & c_{34}+c_{16}+c_{17} & c_{34}+c_{36}+c_{37} & c_{14}+c_{16}+c_{17} \\ c_{18}+c_{19}+c_{1,10} & c_{18}+c_{19}+c_{1,10} & c_{18}+c_{19}+c_{1,10} & c_{38}+c_{39}+c_{3,10} \end{bmatrix}$$

$$P_2 = \begin{bmatrix} c_{31}+c_{32}+c_{33} & c_{33}+c_{24}+c_{25} & c_{24}+c_{26}+c_{27} & c_{28}+c_{29}+c_{2,10} \\ c_{21}+c_{22}+c_{33} & c_{33}+c_{34}+c_{35} & c_{34}+c_{26}+c_{27} & c_{28}+c_{29}+c_{2,10} \\ c_{21}+c_{22}+c_{23} & c_{23}+c_{34}+c_{25} & c_{34}+c_{36}+c_{37} & c_{28}+c_{29}+c_{2,10} \\ c_{21}+c_{22}+c_{23} & c_{23}+c_{24}+c_{25} & c_{24}+c_{26}+c_{27} & c_{38}+c_{39}+c_{3,10} \end{bmatrix}.$$

A weighted congestion game

By the concept of weighted congestion game, the relative payoff matrix is $R = w_2 P_2 - w_1 P_1$. So, the game is a potential game if and only if

$$B_4 R B_4^{\mathrm{T}} = 0,$$

which is simplified as the following equations:

$$\begin{aligned}
&w_2(c_{31} + c_{32} - c_{21} - c_{22}) - w_1(c_{31} + c_{32} - c_{11} - c_{12}) = 0,\\
&w_2(c_{33} - c_{23}) - w_1(c_{33} - c_{13}) = 0,\\
&w_2(c_{34} - c_{24}) - w_1(c_{34} - c_{14}) = 0,\\
&w_2(c_{35} - c_{25}) - w_1(c_{35} - c_{15}) = 0,\\
&w_2(c_{36} + c_{37} - c_{26} - c_{27}) - w_1(c_{36} + c_{37} - c_{16} - c_{17}) = 0,\\
&w_2(c_{38} + c_{39} + c_{3,10} - c_{28} - c_{29} - c_{2,10})\\
&\qquad -w_1(c_{38} + c_{39} + c_{3,10} - c_{18} - c_{19} - c_{1,10}) = 0.
\end{aligned}$$

Conclusions

1. Based on the STP, a finite game can be expressed as a **payoff matrix**.

2. A finite potential game is just like a **potential vector field** (**conservative field**).

3. A finite game is a potential game if and only if its **potential equation** has a solution.

4. The **minimum number** of linear equations for verifying potential games can be obtained.

5. Based on STP, **linear spaces of games** and **congestion games** can be considered.

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Many Thanks for Your Attention!