Evolutionary Games and Networked Evolutionary Games

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Initiation of Evolutionary Games (EGs)

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The Logic of Animal Conflict

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Conflicts between animals of the same species usually are of "limited war" type, not causing serious injury. This is often explained as due to group or species selection for behaviour benefiting the species rather than individuals. Game theory and computer simulation analyses show, however, that a "limited war" strategy benefits individual animals as well as the species.

In a typical combat between two male animals of the same species, the winner gains mates, dominance rights, desirable territory, or other advantages that will tend toward transmitting its genes to future generations at higher frequencies than the loser's genes. Consequently, one might expect that natural selection would develop maximally effective weapons and fighting styles for a "total war" strategy of battles between males to the death. But instead, intraspecific conflicts are usually of a "limited war" type, involving inefficient weapons or ritualized tactics that seldom cause serious injury to either contestant. For example, in many snake species the males fight each other by wrestling without using their fangs[1,2]. In mule deer (*Odocolleus hemionus*) the bucks fight furiously but harmlessly by crashing or pushing antlers against antlers, while they refrain from attacking when an opponent turns away, exposing the unprotected side of its body[3]. And in the Arabian oryx (*Oryx leucoryx*) the extremely long, backward pointing horns are so inefficient for combat that in order for two males to fight they are forced to kneel down with their heads between their knees to direct their horns forward[4]. (For additional examples, see Collins[5], Darwin[4], Hingston[6], Huxley *et al.*[7], Lorenz[8] and Wynne-Edwards[9].)

How can one explain such oddities as snakes that wrestle with each other, deer that refuse to strike "foul blows", and antelope that kneel down to fight?

The accepted explanation for the conventional nature of contests is that if no conventional methods existed, many individuals would be injured, and this would militate against the survival of the species (see, for example, Huxley[7]). The difficulty with this type of explanation is that it appears to assume the operation of "group selection". Although one cannot rule out group selection as an agent producing adaptations, it is only likely to be effective in rather special circumstances[10-12]. Consequently it seems to us that group selection cannot by itself account for the complex anatomical and behavioural adaptations for limited conflict found in so many species, but there must also be individual selection for these, which means that a "limited war" strategy must be differentially advantageous for individuals.

We consider simple formal models of conflict situations, and ask what strategy will be favoured under individual selection. We first consider conflict in species possessing offensive weapons capable of inflicting serious injury on other members of the species. Then we consider conflict in species where serious injury is impossible, so that victory goes to the contestant who fights longest. For each model, we seek a strategy that will be stable under natural selection; that is, we seek an "evolutionarily stable strategy" or ESS. The concept of an ESS is fundamental to our argument; it has been derived in part from the theory of games, and in part from the work of MacArthur[13] and of Hamilton[14] on the evolution of the sex ratio. Roughly, an ESS is a strategy such that, if most of the members of a population adopt it, there is no "mutant" strategy that would give higher reproductive fitness.

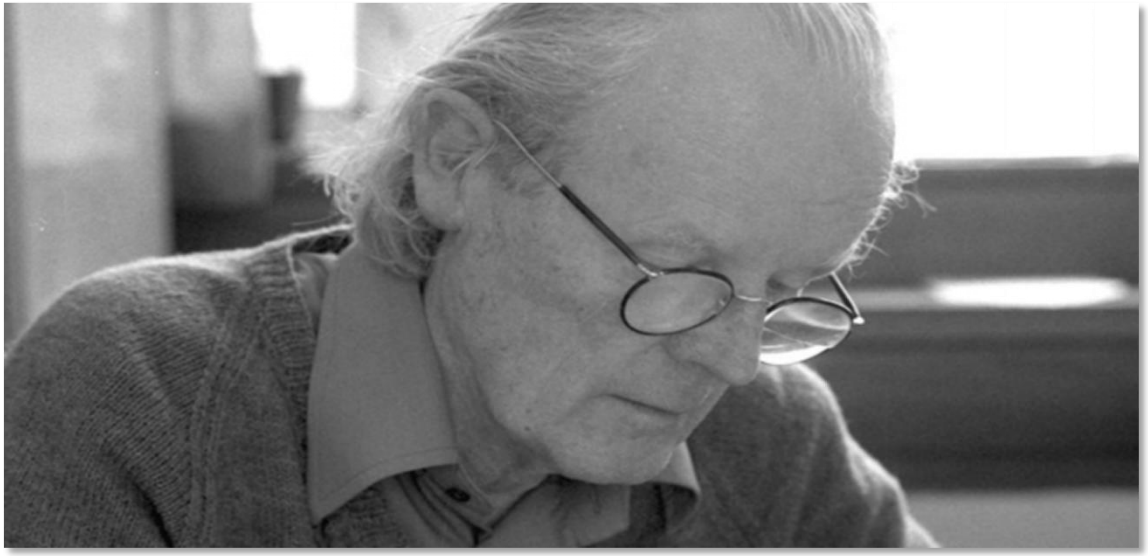
A Computer Model

A main reason for using computer simulation was to test whether it is possible even in theory for individual selection to account for "limited war" behaviour.

We consider a species that possesses offensive weapons capable of inflicting serious injuries. We assume that there are two categories of conflict tactics: "conventional" tactics, *C*, which are unlikely to cause serious injury, and "dangerous" tactics, *D*, which are likely to injure the opponent seriously if they are employed for long. (Thus in the snake example, wrestling involves *C* tactics and use of fangs would be *D* tactics. In many species, *C* tactics are limited to threat displays at a distance, without any physical fighting. We consider a conflict between two individuals to consist of a series of alternate "moves". At each move, a contestant can employ *C* or *D* tactics, or retreat, *R*. If a contestant employs *D* tactics, there is a fixed probability that his opponent will be seriously injured: a contestant who is seriously injured always retreats. If a contestant retreats, the contest is at an end and his opponent is the winner. A possible conflict between contestants *A* and *B* can be represented in this way:

A's move *C C C C C C C C C C C D C C C C C C C D*
B's move *C C C C C C C C C C C D C C C C C C C R*

If a contestant plays *D* on the first move of a contest, or plays *D* in response to *C* by his opponent, this is called a "probe" or a "provocation". A probe made after the opening move is said to "escalate" a contest from *C* to *D* level. A contestant who plays *D* in reply to a probe is said to "retaliate". In the example shown above, *A* probes on his twelfth and twentieth moves; *B* retaliates after the first probe, but retreats after the second, leaving *A* the winner. At the end of a contest there are "pay-offs" to each contestant. The pay-offs are taken as measures of the contribution the contest has made to the reproductive success of the individual. They take account of three factors: the advantages of winning as compared with losing, the disadvantage of being seriously injured, and the disadvantage of wasting time and energy in the contest.



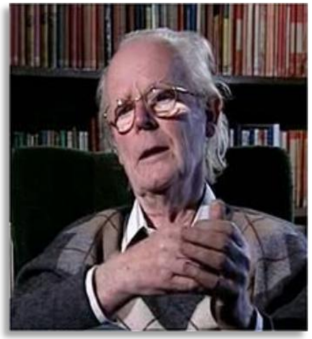
演化博弈论是将博弈论运用到生物学中群体生命的演化问题，或用演化理论来发展博弈论。演化博弈论起源于1973年，由 J. M. Smith 和 G. R. Price 提出。

[1] J. M. Smith, G. R. Price, The logic of animal conflict, Nature, Vol. 10, No. 5427, 15-18, 1973.

演化博弈论起源于一个具体的**生物学**问题：**如何解释动物在冲突情景中的仪式化行为**。

为什么有些动物在争夺资源中表现的非常“绅士”或“淑女”？

Tinbergen 提出这种行为是为了物种的利益。



Maynard Smith 无法看出Tinbergen的推理如何与达尔文的思想相匹配。

Motivation of EGs

在经典博弈论中，参与博弈的玩家常常被假定是完全理性的，且具有完全信息。但在实际应用中，玩家的完全理性和完全信息假设常常很难满足。

完全信息

每个玩家都了解其他玩家的收益函数的博弈

完全理性

由具备完全理性的玩家的所进行的博弈

✖ ✖

Maynard Smith意识到在演化博弈论中并不需要每一个玩家都理性行事，通过演化可以检验不同策略在环境中的生存和复制能力。

与经典博弈论不同，演化博弈理论放弃了上述两个关于参与玩家的基本假设，转而利用生物进化论中的自然选择、突变等机制，来分析和预测参与玩家的策略演化过程和动态过程。

Some Representative Games

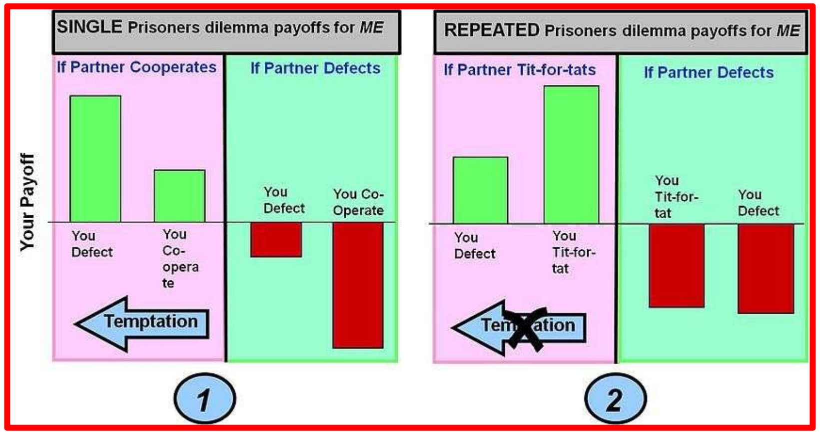


Figure 1: Prisoner's dilemma

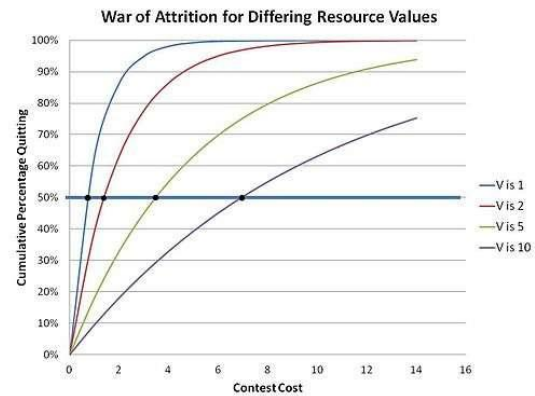


Figure 2: War of attrition

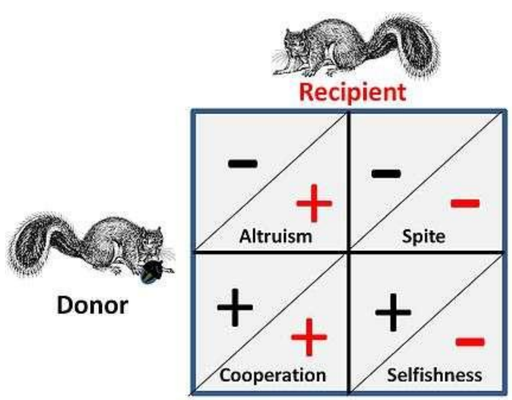


Figure 3: Strategic alternatives in social behaviour

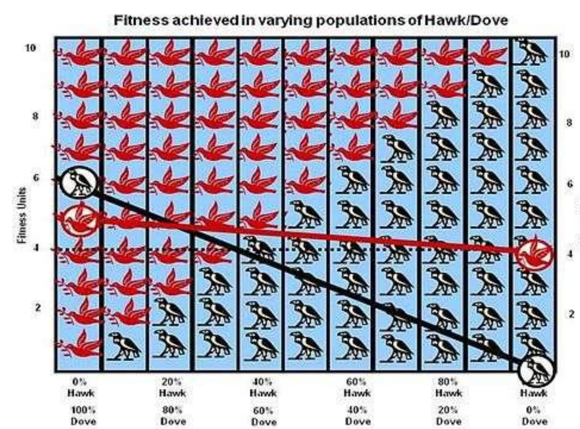


Figure 4: Hawk-dove

Development of EGs

1950s

Alchian (1950) 建议在经济分析中用**自然选择**的概念代替利润最大化概念。Nash (1951) 的"**群体行为解释**"是包含较完整演化博弈思想的最早理论成果[1]。

1970s

Smith发表**《The logic of animal conflict》**，这**标志着演化博弈论的诞生**。Smith和Price提出演化博弈论中的**基本概念**"**演化稳定策略**"。1978年，生态学家Taylor和Jonker提出了演化博弈理论的基本动态概念——**复制动态**。

1980s

经济学家把**演化博弈论**引入到**经济学领域**，用于分析社会制度变迁、产业演化以及股票市场等，同时对演化博弈论的研究也开始由对称博弈向**非对称博弈**深入。

[2] J. F. Nash, Non-cooperative games, Annals of Mathematics, No. 54, 286-295, 1951.

1990s

演化博弈论的发展进入一个新的阶段。Weibull (1995) 比较系统、完整地总结了演化博弈论，其中包含了一些最新的理论研究成果。

21世纪

演化博弈的发展出现了一些新的思路，对演化稳定策略和**合作演化博弈**的研究不断深入，学者开始关注带有**随机因素**影响的演化过程。进入21世纪以来，国内的学者开始关注演化博弈论，也做出了大量研究。

[3] K. Basu K, J. Weibull, Strategy subsets closed under rational behavior, Economics Letters, Vol. 36, No. 2, 141-146, 1991.
[4] J. Bengtsson, J. Ahnstrm, A. Weibull, The effects of organic agriculture on biodiversity and abundance: A meta-analysis, Journal of Applied Ecology, Vol. 42, No. 2, 261-269, 2010.
[5] B. Jin, H. Li, W. Yan and M. Cao, Distributed model predictive control and optimization for linear systems with global constraints and time-varying communication, IEEE Transactions on Automatic Control, Vol. 66, No. 7, 3393-3400, 2021.

EGs Model

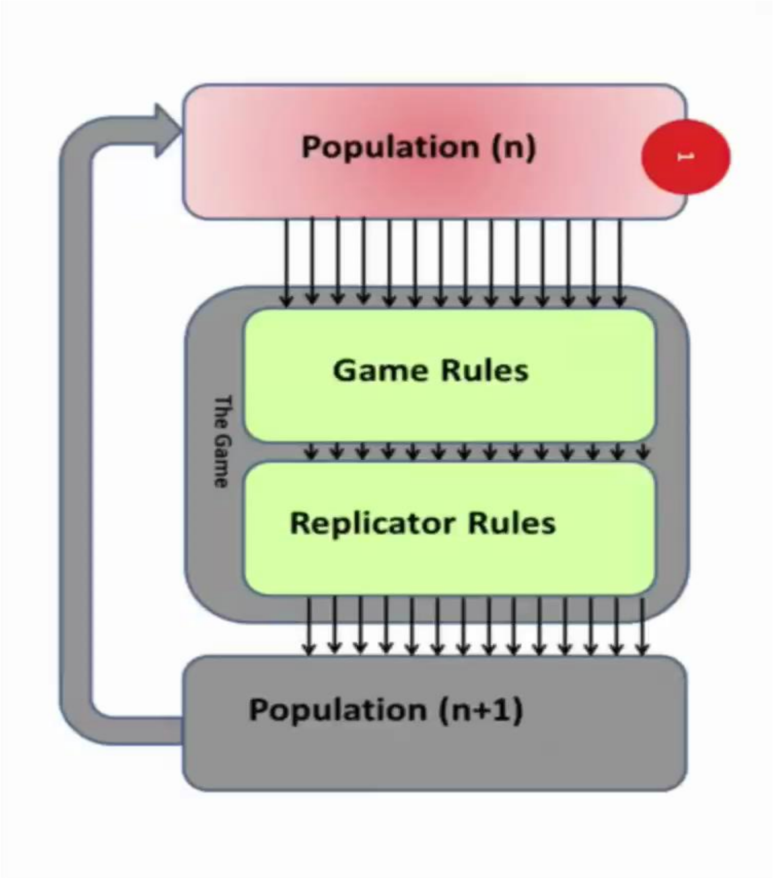


Figure 5: 演化博弈模型通过采用**进化过程**的系统模型，将达尔文机制转化为数学形式，该系统模型由三个主要组成部分——**种群**、**博弈**和**复制动力学**。

[6] J. M. Smith, Evolution and the theory of games, American Scientist, Vol. 64, No. 1, 41-45, 1976.

Evolutionarily Stable Strategy (ESS)

J. M. Smith 和 G. R. Price 提出了**演化稳定策略**的基本概念，该**均衡概念**的提出使得演化博弈理论的有了明确的方向，为其进一步发展奠定了坚实的基础。

To answer this question, we need a more precise definition of an ESS. We define $E_J(I)$ as the expected pay-off to I played against J. Then I is an ESS if, for all J, $E_I(I) > E_I(J)$; if for any strategy J, $E_I(I) = E_I(J)$, then evolutionary stability requires that $E_J(I) > E_J(J)$. The relevance of the latter condition is as follows. If in a population adopting strategy I a mutant J arises whose expectation against I is the same as I's expectation against itself, then J will increase by genetic drift until meetings between two J's becomes a common event.

[1] J. M. Smith, G. R. Price, The logic of animal conflict, Nature, Vol. 246, No. 5427, 15-18, 1973.

Repeated Game

演化博弈中应用较为广泛的基本动态方程是**复制者动态方程**，生态学家Taylor和Jonker在考察生态演化现象时首次提出演化博弈理论的基本动态概念——**复制者动态 (repeated dynamic)**，这是演化博弈理论发展的又一座里程碑。

[7] R. Taylor, L. Jonker, Evolutionarily stable strategies and game dynamics, Mathematical Biosciences, Vol. 40, No. 1-2, 145-156, 1978.

Repeated Game

在博弈学习框架中，同一个博弈被假定**重复多次**，称为重复博弈。玩家利用在重复博弈中获得的信息，不断更新自己的策略。具体地，考虑一个**离散时间**的**重复博弈** $G=\left\{N,\{S_i: i\in N\},\{c_i: i\in N\}\right\}$. 在每个时间步 t， 每个玩家 $i\in N$根据当前的自身策略 $s_i(t)\in S_i(t)$、其他玩家的策略以及在博弈中的收益 $\pi_i(t)=c_i(s(t))$, 按照一定的学习规则更新自己的策略。

一般形式的学习规则表示如下：

$$s_i(t+1)=\mathcal{F}_i\left(\prod_{k=0}^{t} s_i(k);\prod_{k=0}^{t} s_{-i}(k);c_i\right), \tag{1}$$

其中, $\mathcal{F}_i$ 可以是确定性函数或者随机函数。

[8] P. H. Young, Strategic Learning and Its Limits, Oxford University Press, 2004.

Repeated Game

在上述一般形式的学习机制中，要求每个玩家具有无限的记忆功能，但更常见的情形是，每个玩家只**具有一步记忆功能。** 在这种情况下，上述学习规则应改为：

$$s_i(t+1) = \mathcal{F}_i\Big(s_i(t); s_{-i}(t); c_i\Big). \tag{2}$$

根据每个玩家更新策略的时序，可将重复博弈分为**同步学习**、**异步学习**、**顺序学习**和**随机时序学习**等类型.

- 同步学习(synchronous learning): 在每个时刻 t, 所有玩家根据对应的学习规则, 同时更新自身的策略.
- 异步学习(asynchronous learning): 在每个时刻 t, 只有一部分玩家更新自己的策略，其他玩家保持其原来的策略不变. 例如, 每个个体玩家 $i \in N$ 以概率 $p_i \in (0,1)$ 更新自己的策略, 以概率 $1-p_i$ 保持自己原来的策略. 这种更新方式属于异步学习.
- 顺序学习(sequential learning): 玩家依照指定的次序依次更新自己的策略. 每个时刻 t, 只有一个玩家更新自身策略, 其他玩家保持原来策略不变.
- 随机时序学习(random-timing learning): 每个时刻 t 按照一定的概率 $q_i \in (0,1)$ 选择一个玩家 $i \in N$ 更新自己的策略, 其中 $\sum\limits_{i\in N} q_i = 1$.

Strategy Profile Dynamics (SPD)

因为**收益信息**可以**间接**地由**玩家策略**得到，所以一个n 人演化博弈可以表示为如下形式：

$$\begin{cases} s_1(t+1) = f_1(s(t), s(t-1), \cdots, s(0)) \\ s_2(t+1) = f_2(s(t), s(t-1), \cdots, s(0)) \\ \qquad \vdots \\ s_n(t+1) = f_n(s(t), s(t-1), \cdots, s(0)), \end{cases}$$

其中, $s(l) = (s_1(l), \cdots, s_n(l))$ 表示每个玩家在 t 时刻的策略. 注意: f_i, $i \in N$ 可能是一种概率映射, 这意味着玩家 i 使用的是混合策略.

[9] H. Qi, Y. Wang, T. Liu, D. Cheng, Vector space structure of finite evolutionary games and its application to strategy profile convergence, Journal of Systems Science & Complexity, Vol. 29, 602-628, 2016.

假设每个玩家只具有一步记忆功能，即其下一时刻的策略仅仅**依赖于当下的策略**（**马尔可夫决策过程**），演化方程变为：

$$
\begin{cases}
s_1(t+1) = f_1(s_1(t), s_2(t), \cdots, s_n(t)) \\
s_2(t+1) = f_2(s_1(t), s_2(t), \cdots, s_n(t)) \\
\quad\vdots \\
s_n(t+1) = f_n(s_1(t), s_2(t), \cdots, s_n(t)).
\end{cases}
\tag{3}
$$

演化博弈的性质是由其**策略局势动态**唯一决定的！

Strategy Updating Rule (SUR)

短视最优响应 Mypoic best response asjustment (MBRA)

Construct a set of optimal response set of strategies at t as

$$O_i(t) = \text{argmax}_{s_i \in S_i} c_i(s_i, s^{-i}(t)).$$

Then

(i) (Case 1) If $x_i(t) \in O_i(t)$, then $x_i(t+1) = x_i(t)$;

(ii) (Case 2) If $x_i(t) \notin O_i(t)$, then

- Deterministic Model (MBRA-D): Choose smallest j, such that $s_j \in O_i(t)$, and set $x_i(t+1) = s_j$.
- Stochastic Model (MBRA-S): Choose any $j \in O_i$, with equal probability $p = 1/|O_i|$.

演化博弈的**策略局势动态**由**策略更新规则**决定

无条件模仿Unconditional Imitation

II-*I* :Unconditional Imitation with Fixed Priority

The best strategy from strategies of players $\{j \mid j \in N\}$ at time t is selected as the strategy of player i at time $t+1$, denoted by $x_i(t+1)$. Precisely, if

$$j^* = \text{argmax}_{j \in N} \; c_j(x(t))$$

then

$$x_i(t+1) = x_{j^*}(t).$$

When the players with the best payoff are not unique, say

$$\text{argmax}_{j \in N} \; c_j(x(t)) := \{j_1^*, \ldots, j_r^*\}$$

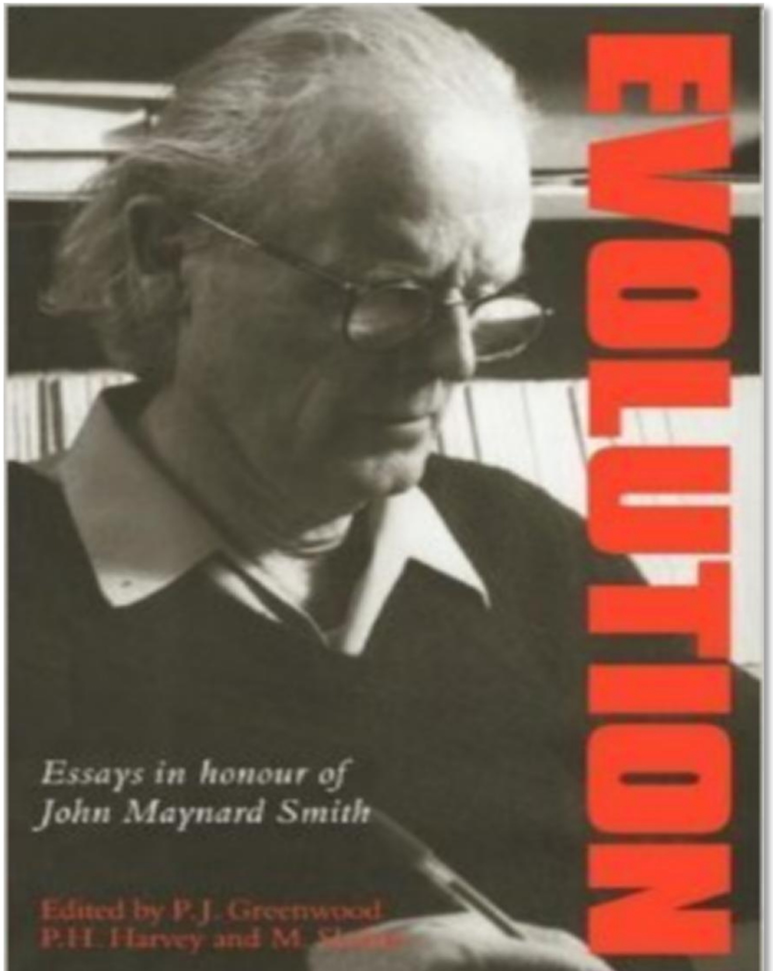
无条件模仿 Unconditional Imitation

II-*II* :Unconditional imitation with equal probability for best strategies. When the best payoff player is unique, it is the same as Π-I. When the players with best payoff are not unique, say, as in (16), then we randomly choose one with equal probability. That is

$$x_i(t+1) = x_{j_\mu^*}(t), \quad \text{with probability} \quad p_\mu^i = \frac{1}{r}$$

$$\mu = 1, \ldots, r.$$

This method leads to a probabilistic k-valued dynamics.



一盏司代数学比一吨口头争论
更有价值

"An ounce of algebra is worth a ton of verbal argument"

——J.B.S. Haldane

(as quoted by John Maynard Smith)

Semi-tensor Product (STP)

有限演化博弈就是有限个玩家策略选择的动态优化过程，每个玩家都有**有限个策略**可以选择。所以，当博弈演化依赖有限历史信息(特别是只依赖上一时刻信息)时，**有限博弈的动态过程**可以用一个**有限值逻辑系统**进行描述。

半张量积方法对博弈论的研究具有天生的**优越性**，极具发展潜力

[10] D. Cheng, An Introduction to Semi-tensor Product of Matrices and Its Applications. World Scientific, 2012.

Semi-tensor Product (STP)

系 统 科 学 与 数 学
J. Sys. Sci. & Math. Scis.
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演化博弈与逻辑动态系统的优化控制*

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摘要 探讨演化博弈与逻辑动态系统的优化控制的关系. 主要内容包括三个方面: 1) 讨论基于演化博弈的逻辑动态 (控制) 系统的建模, 即如何从演化动态博弈导出多值逻辑系统的优化控制问题; 2) 多值逻辑系统在平均收益和带贴现因子的总收益两种性能指标下的优化控制的基本结论与算法; 3) 如何从逻辑动态系统的最优控制导出演化博弈的纳什均衡. 使用的基本工具是矩阵的半张量积, 基本方法是将逻辑动态系统转化为基于矩阵的离散时间动态系统和博弈策略的矩阵表示.

关键词 演化博弈, 逻辑动态系统, 最优控制, 纳什均衡, 矩阵半张量积.

MR(2000) 主题分类号 91A06

程代展教授探讨了演化博弈与逻辑动态系统的优化控制之间的关系，利用矩阵半张量积将演化博弈的动态模型转化为代数形式，分别给出了两种不同性能指标下优化控制的基本结论与算法。

[11] 程代展，赵寅，徐听听.演化博弈与逻辑动态系统的优化控制, 系统科学与数学， Vol.32, No.10, 1226-1238, 2012.

Semi-tensor Product (STP)

STP通过将**有限演化博弈动力学**建模为严格的**逻辑网络**，给出了一种新的博弈表达方式:

Identify each strategy $s_i = j \in S_i$ by the vector form $\delta_{k_i}^j$, $i \in N$. Then, $S_i \sim \Delta_{k_i}$. The strategy profile $s = (s_1, s_2, \cdots, s_n)$ is expressed as the vector form

$$s = \ltimes_{i=1}^n s_i \in \Delta_k. \qquad (4)$$

The payoff function c_i is expressed as $c_i(s_1, s_2, \cdots, s_n) = V_i^c \ltimes_{j=1}^n s_j$, $i = 1, \cdots, n$, where $V_i^c \in \mathbb{R}^k$ is called the structure vector of c_i. Collecting the structure vector of each player, we obtain the structure vector of G as

$$V_G^c = [V_1^c \; V_2^c \; \cdots \; V_n^c] \in \mathbb{R}^{nk}. \qquad (5)$$

Semi-tensor Product (STP)

Based on STP, convert (3) into the **algebraic form**

Denote $x_i(t) \in \Delta_{k_i}$, $i = 1, \cdots, n$, we have

$$x_i(t+1) = M_i(t), \ \ i = 1, \cdots, n, \qquad (6)$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t)$, $M_i \in \mathcal{L}_{k_i \times k}$ is the structure matrix of f_i, $i = 1, \cdots, n$.

Multiplying these equations together yields the following algebraic form:

$$x(t+1) = Mx(t), \qquad (7)$$

where $M := M_1 * \cdots * M_n \in \mathcal{L}_{k \times k}$.

Example 1

We give a numerical example to illustrate **the vector space of finite games** and how to **use the SUR to determine the strategy profile dynamics**.

Example 1: A game G has 3 players. Player 1 and player 3 have 2 strategies, and player 2 has 3 strategies. Then we have $N=\{1,2,3\}$, $S_1=\{1,2\}$, $S_2=\{1,2,3\}$, $S_3=\{1,2\}$. That is, $n=3$, $k_1=k_3=2$, $k_2=3$, and $k=2\cdot 3\cdot 2=12$. So $G\in\mathcal{G}_{[3;2,3,2]}$.

As a vector, $V_G^c\in\mathcal{V}^{\mathcal{G}}$, which has dimension $nk=36$. Next, assume Table 1 is the payoff matrix of G.

Table 1 Payoff matrix of Example 3.8

c	s											
	111	112	121	122	131	132	211	212	221	222	231	232
c_1	1	−1	2	1	0	4	3	−2	3	0	−3	−4
c_2	−1	2	−3	−2	3	5	3	3	−1	−1	2	−1
c_3	0	5	−2	2	−1	4	2	4	−3	−2	3	2

Then, we have

$$V_1^c = [1, -1, 2, 1, 0, 4, 3, -2, 3, 0, -3, -4],$$
$$V_2^c = [-1, 2, -3, -2, 3, 5, 3, 3, -1, -1, 2, -1],$$
$$V_3^c = [0, 5, -2, 2, -1, 4, 2, 4, -3, -2, 3, 2],$$

and

$$V_G^c = [V_1^c, V_2^c, V_3^c].$$

如何使用**保持算子**求解后两个方程?

Sequential MBRA

Assume **player 1** is chosen to update its strategy. Then we have

$$x_1(t+1) = f_1(x_1(t), x_2(t), x_3(t)) = \delta_2[2,1,2,1,1,1,2,1,2,1,1,1]x(t),$$
$$x_2(t+1) = x_2(t) = \delta_3[1,1,2,2,3,3,1,1,2,2,3,3]x(t),$$
$$x_3(t+1) = x_3(t) = \delta_2[1,2,1,2,1,2,1,2,1,2,1,2]x(t),$$

where $x(t) = \ltimes_{i=1}^3 x_i(t)$.

$$x(t+1) = M_1 x(t),$$

where

$$M_1 = \delta_{12}[7,2,9,4,5,6,7,2,9,4,5,6].$$

Similarly, if **player 2** is chosen to update its strategy, then we have

$$x(t+1) = M_2 x(t),$$

where

$$M_2 = \delta_{12}[5,6,5,6,5,6,7,8,7,8,7,8].$$

If **player 3** is chosen to update its strategy, then we have

$$x(t+1) = M_3 x(t),$$

where

$$M_3 = \delta_{12}[2,2,4,4,6,6,8,8,10,10,11,11].$$

Then in **periodic type** we have

$$
\begin{cases}
x(t+1) = M_1 x(t), & t = 3k, \\
x(t+1) = M_2 x(t), & t = 3k+1, \qquad k = 0, 1, \cdots . \\
x(t+1) = M_3 x(t), & t = 3k+2,
\end{cases}
$$

In **random-timing type** we have

$$
x(t+1) = Mx(t), \quad M = \frac{1}{3}(M_1 + M_2 + M_3) = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&2&0&0&0&0&0&1&0&0\\
1&0&1&0&2&0&0&0&0&0&1&0\\
0&1&0&1&1&3&0&0&0&0&0&1\\
1&0&0&0&0&0&2&0&1&0&1&0\\
0&0&0&0&0&0&1&2&0&1&0&1\\
0&0&1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}.
$$

基于STP方法，目前演化博弈论方向已经有了一些初步的结果

Optimization

下述文献研究了**EG**的**策略最优**和**最优控制**等相关问题

[12] 程代展, 赵寅, 徐听听. 演化博弈与逻辑动态系统的优化控制, 系统科学与数学, Vol. 32, No.10, 1226-1238, 2012.

[13] G. Zhao, Y. Wang, H. Li, A matrix approach to modeling and **optimization** for dynamic games with **random entrance**, Applied Mathematics and Computation, No. 290, 9-20, 2016.

[12] 程代展，赵寅，徐听听.演化博弈与逻辑动态系统的优化控制, 系统科学与数学, Vol. 32, No.10, 1226-1238, 2012.

对于动态博弈，本文考虑两类收益函数

1) 平均收益

$$J_i = \overline{\lim_{T\to\infty}} \frac{1}{T}\sum_{t=1}^{T} c_i(x_1(t), x_2(t), \cdots, x_n(t)), \quad i = 1, 2, \cdots, n.$$

2) (带贴现因子的) 总收益

$$J_i = \sum_{t=1}^{\infty} \lambda^t c_i(x(t), u(t)), \quad i = 1, 2, \cdots, n,$$

这里 $0 < \lambda < 1$ 称贴现因子 [4].

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), u_2(t), \cdots, u_m(t)), \\ x_2(t+1) = f_2(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), u_2(t), \cdots, u_m(t)), \\ \quad \vdots \\ x_n(t+1) = f_n(x_1(t), x_2(t), \cdots, x_n(t), u_1(t), u_2(t), \cdots, u_m(t)), \end{cases} \tag{7}$$

这里 $x_i \in \mathcal{D}_{k_i}$, $u_j \in \mathcal{D}_{d_j}$.

定理 9[17–18] 考虑系统 (7) 在 1) 平均收益；或 2) 带贴现因子的总收益，下的最优控制．则

1) 一组最优策略及相应轨线在状态—控制空间中收敛于周期轨道 (即在有限步后进入周期轨道).

2) 最优策略 $(u^*(t))$ 满足

$$u^*(t+1) = g(x(t), u(t)) = L_g u(t) x(t),$$

这里， $L_g \in \mathcal{L}_{p \times pq}$.

基于上述结果，寻优算法大致可分为两种

1) 方法一：通过周期轨道寻找最优解.

2) 方法二：在策略空间直接寻找最优控制.

Convergence

下述文献研究了**EG**的**稳定性**和**收敛性**，以及Nash**均衡点**等问题

[14] H. Qi, Y. Wang, T. Liu, D Cheng, Vector space structure of finite evolutionary games and its application to strategy profile **convergence**, Journal of Systems Science & Complexity, Vol. 29, 602-628, 2016.

[15] D. Cheng, J. Liu, Lyapunov function approach to **convergence** of finite evolutionary games, Proceeding of the 11th World Congress on Intelligent Control and Automation, 3040-3045, 2014.

[16] Y. Wang, D. Cheng, Dynamics and **stability** for a class of evolutionary games with time delays in strategies, Science China Information Sciences, Vol. 59, No. 9, 092209, 2016.

[17] Y. Wu, M. Toyoda, T. Shen, **Linear dynamic games** with polytope strategy sets. IET Control Theory and Applications, Vol. 11, No.13, 2146-2151, 2017.

[18] D. Cheng, H. Qi, Y. Wang, T. Liu, On **convergence** of evolutionary games, Proceedings of the 33rd Chinese Control Conference, 5539-5545, 2014.

[19] X. Zhang, D. Z. Cheng, **Profile-dynamic** based fictitious play, Science China Information Sciences, Vol. 64. No. 6, 169202, 2021.

[14] H. Qi, Y. Wang, T. Liu, D Cheng, Vector space structure of finite evolutionary games and its application to strategy profile convergence, Journal of Systems Science & Complexity, Vol. 29, 602-628, 2016.

The **Lyapunov function** of EGs is defined in [14] and its application to **the convergence of EGs** is presented.

Definition 8.1 Let $G \in \mathcal{G}_{[n;k_1,k_2,\cdots,k_n]}$, $k := \prod_{i=1}^{n} k_i$.

1) A pseudo-logical function $\psi : \Delta_k \to \mathbb{R}$ is called a Lyapunov function of G if

$$\psi(x(t+1)) - \psi(x(t)) \geq 0, \quad t \geq 0,$$

and $\psi(x(t+1)) = \psi(x(t))$ implies $x(t+1) = x(t)$.

2) When the mixed strategies are allowed, in the above definition ψ should be replaced by its expected value, i.e., $E\psi : \Upsilon_k \to \mathbb{R}$ with

$$E\psi(x(t+1)) - E\psi(x(t)) \geq 0, \quad t \geq 0,$$

and $E\psi(x(t+1)) = E\psi(x(t))$ implies $Ex(t+1) = Ex(t)$.

Theorem 8.2 *An EG will converge to an equilibrium if there is a Lyapunov function.*

Theorem 8.4 *Given a deterministic game G with its strategy profile dynamics* (61) *with* $T = \delta_k[i_1, i_2, \cdots, i_k]$. *G has a Lyapunov function if and only if*

(i)

$$a_{i_j} \geq a_j, \quad j = 1, 2, \cdots, k$$

has solution a_j, $j = 1, 2, \cdots, k$;

(ii) $a_{i_j} = a_j$ *implies* $i_j = j$.

Moreover, in [14] the **near potential function** for an EG is defined, and it is proved that if the near potential function of an EG is a **Lyapunov function**, the EG will **converge** to a **pure Nash equilibrium**.

Stochastic

[20] X. Ding, H. Li, Q. Yang, Y. Zhou, F. E. Ahmed Alsaedi, **Stochastic** stability and stabilization of n-person random evolutionary Boolean games, Applied Mathematics and Computation, No. 306, 1-12, 2017.

[21] H. Li, X. Ding, F. E. Ahmed Alsaedi, **Stochastic** set stabilization of n-person random evolutionary Boolean games and its applications, IET Control Theory & Applications, Vol. 11, No. 13, 2152-2160, 2017.

[22] X. Ding, H. Li, F. E. Ahmed Alsaedi, Regulation of game result for n-person **random** evolutionary Boolean games, Asian Journal of Control, No. 22, 2353-2362, 2020.

[23] X. Ding, H. Li, Optimal control of **random** evolutionary Boolean games, International Journal of Control, No. 306, 1-12, 2019.

针对策略局势动态中带有随机干扰的情形，上述文献系统研究了**随机演化布尔博弈**的**稳定性、集合镇定、调节和最优控制**等问题。

Problem Formulation

第一节提到的模型是一种**非常一般化**的模型，它对于参与博弈的玩家之间的结构不做任何具体的假设，每个参与玩家的收益可能与其它所有玩家的策略相关。

演化博弈注重考虑的是个体的**局部交互规则**对整个博弈动态的影响。

当考虑玩家在**复杂网络**上进行演化博弈时，每个玩家通过其邻近玩家对整个博弈动态产生影响。

为了刻画网络上局部交互的博弈关系，网络演化博弈被提出

[24] M. O. Jackson, Y. Zenou, Games on Networks, Handbook of Game Theory with Economic Applications, No. 4, 95-163, 2015.
[25] 王龙, 伏锋, 陈小杰, 王靖, 李卓政, 谢广明, 楚天广, 复杂网络上的演化博弈, 智能系统学报, Vol. 2, No. 2, 1-10, 2007.

Network Graph

网络普遍存在于现实生活和自然界中，比较常见的有交通网络、因特网、人际关系网络、河流网、基因网络、神经网络、食物网等。网络作为一种模型，可以用来描述系统中的对象（节点）与对象之间的关系（边）。在实际网络中，节点间的局部交互方式往往比较复杂，具有一些典型的拓扑特征，这类系统的拓扑结构往往使用复杂网络模型来刻画。

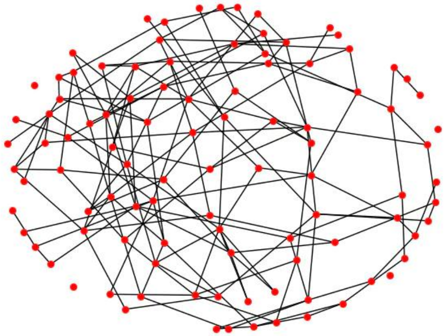

Figure 6: Erdos-Renyi 随机图

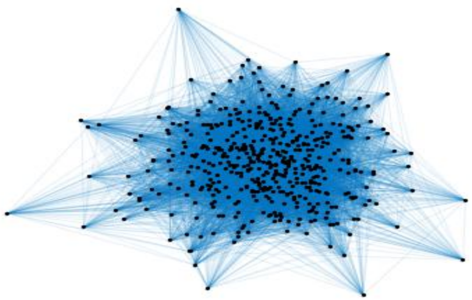

Figure 7: Watts-Strogatz 小世界网络

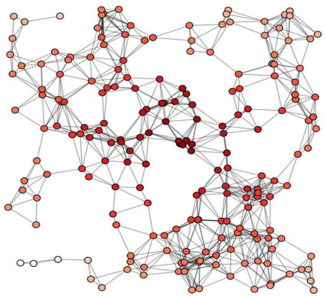

Figure 8: 随机几何图

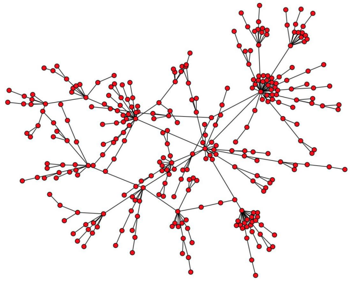

Figure 9: Barabasi-Albert 无标度网络

[26] D. M. Boyd, N. B. Ellison, Social network sites: definition, history, and scholarship, IEEE Engineering Management Review, Vol. 38, No. 3, 16-31, 2010.
[27] 吕金虎，谭少林，复杂网络上的博弈及其演化动力学，高等教育出版社，2019.

Network Evolutionary Games (NEGs)

“囚徒困境”可以代表现实中的很多合作现象，然而当把“囚徒困境”放在**完全混合**的环境下时，因为每个玩家都**完全**与其他**个体交互**，所以背叛者的收益永远要高于合作者，因此**合作者**在演化过程中将**被逐渐淘汰掉**。1992 年 Nowak 和 May把“囚徒困境”模型放在了**空间二维格子上**让其演化，惊喜地发现空间网络不仅可以**促进合作行为**的产生，而且还产生了美妙的类似分形的空间万花筒和合作大爆炸现象[28]。

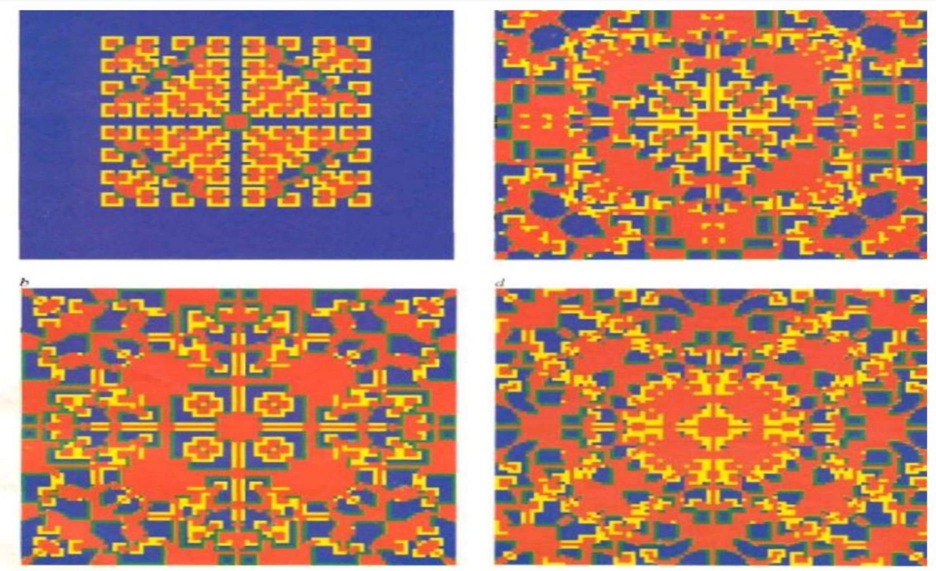

Figure 10: 二维格子上的“囚徒困境”演化结果，蓝色为合作者，红色为背叛者，按时间顺序依次是：左上→右上→左下→右下

[28] M. A. Nowak, R. M. May, Evolutionary games and spatial chaos, Nature, Vol. 359, No.6398, 826-829, 1992.
[29] M. Perc, J. G. Gardeñes, A. Szolnoki, L. M. Floria, Y. Moreno, Evolutionary dynamics of group interactions on structured populations: A review,” J. R. Soc. Inter., Vol. 10, 20120997, 2013.

Development of NEGs

近些年，随着复杂网络理论的快速发展，NEGs已经成为学者们研究的热点问题，且被广泛应用到社会、生物、经济等各个领域[30, 31]。

在网络演化博弈研究的**前期阶段**，研究方向大多集中于**给定静态网络拓扑结构**(如规则网络、小世界网络、无标度网络等)，探讨博弈动力学的演化趋势及结果，研究合作行为产生的机制，以解释或解决一些实际问题。

如：通过生长和偏好连接规则生成的无标度网络为合作行为主导地位的形成提供了充分条件[32]；文献[33]研究得出适当的收益期望水平可以促进合作行为的演化等。

[30] C. Hauert and M. Doebeli, Spatial structure often inhibits the evolution of cooperation in the snowdrift game, Nature, Vol. 428, 643-646, 2004.
[31] M. A. Nowak, R. M. May, Evolutionary games and spatial chaos, Nature, Vol. 359, 826-829, 1992.
[32] F. Santos, J. Pacheco, Scale-free networks provide a unifying framework for the emergence of cooperation, Physical Review Letters, Vol. 95, No. 9, 098104, 2005.
[33] X. Chen, L. Wang, Promotion of cooperation induced by appropriate payoff aspirations in a small-world networked game, Physical Review E, Vol. 77, No. 1, 017103, 2008.
[34] R. Li, J. Yu, J. Lin, Evolution of cooperation in spatial Traveler's Dilemma game, Plos One, Vol. 8, No.3, 1-11, 2013.

受各种外部因素的影响，演化过程中的网络拓扑并非一成不变，单纯研究不同特定网络对博弈动力学的影响已经无法满足实际需求。

这就促使学者将研究方向转向**时变网络拓扑**，考虑交互个体间的网络结构与博弈动力学的协同演化，即在**网络拓扑结构变化**的情况下，研究群体行为与交互结构的**共同涌现**现象。

如：Zimmermann等首先研究了动态网络上的演化博弈[35]；文献[37]研究了基于期望值的个体移动对网络演化囚徒困境的影响；Zhang等人在演化网络下的囚徒困境博弈中提出了消除机制等[38]。

[35] M. Zimmermann, V. Eguiluz, M. San Miguel. Coevolution of dynamical states and interactions in dynamic networks. Physical Review E, Vol. 69, No. 6, 065102, 2004.
[36] M. Zimmermann, V. Eguiluz, Cooperation, social networks, and the emergence of leadership in a prisoner's dilemma with adaptive local interactions, Physical Review E, Vol. 72, No. 5, 1-16, 2005.
[37] Y. Lin, H. Yang, Z. Wu, B. Wang, Promotion of cooperation by aspiration-induced migration. Physica A, Vol. 390, No. 1, 77-82, 2011.
[38] J. Zhang, X. Chen, C. Zhang, et al. Elimination mechanism promotes cooperation in coevolutionary Prisoner's Dilemma games. Physica A, Vol. 389, No. 19, 4081-4086, 2010.

Motivation of Studying NEGs via STP

现有的网络演化博弈的研究工作主要使用计算机仿真[39]，或数值方法[40]

由于**缺少有效的数学工具**，系统地分析网络演化博弈动态过程中的各个玩家的行为是一个非常艰难的工作，深入的研究结果很少。而且，现存的工作多集中在**双策略网络化博弈**问题，如囚徒困境、雪堆博弈等，对于**多策略博弈研究较少**，而多玩家多策略问题广泛存在于像市场经济、电力调配这样的实际系统中，因此值得我们进一步关注。

[39] R. Li, J. Yu, J. Lin, Evolution of cooperation in spatial Traveler's Dilemma game, Plos One, Vol. 8, No. 3, 1-11, 2013.
[40] Y. Achdou, I. Capuzzo-Dolcetta, Mean field games: numerical methods, SIAM Journal on Numerical Analysis, Vol. 48, No. 3, 1136-1162, 2010.

NEGs

考虑一个NEG, 其中每个玩家都理智地与其相邻玩家进行相同的博弈，并假设SUR对所有玩家都是相同的。

正如 [41]中所提到的, 由于缺乏合适的数学工具，**直接分析NEG的动力学是困难的**，目前使用的方法多为基于仿真的分析[41, 42].

[41] L. Wang, F. FU, X. Chen, J. Wang, Z. Li, G. Xie, T. Chu, Evolutionary games on complex networks, CAAI Transactions on Intelligent Systems, Vol. 2, No. 2, 1-9, 2007.
[42] G. Szabo and C. Toke, Evolutionary prisoner’s dilemma game on a square lattice, Phys. Rev. E, Vol. 58, 69-73, 1998.

NEGs Based on STP

We first give a **rigorous definition** of **NEGs**.

Network graph

Fundamental network game (FNG)

Strategy updating rule (SUR)

[43] D. Cheng, F. He, H. Qi, T. Xu, Modeling, analysis and control of networked evolutionary games, IEEE Transactions on Automatic Control, Vol. 60, No. 9, 2402-2415, 2015.

Network Graph

Given a set $N = \{1, 2, \dots, n\}$ and $E \subset N \times N$, (N, E) is called a graph, where N is the set of nodes and E is the set of edges. If $(i, j) \in E$ implies $(j, i) \in E$, the graph is undirected, otherwise, it is directed. Let $N' \subset N$, and $E' = (N' \times N') \cap E$. Then (N', E') (briefly, N') is called a sub-graph of (N, E) (briefly, N).

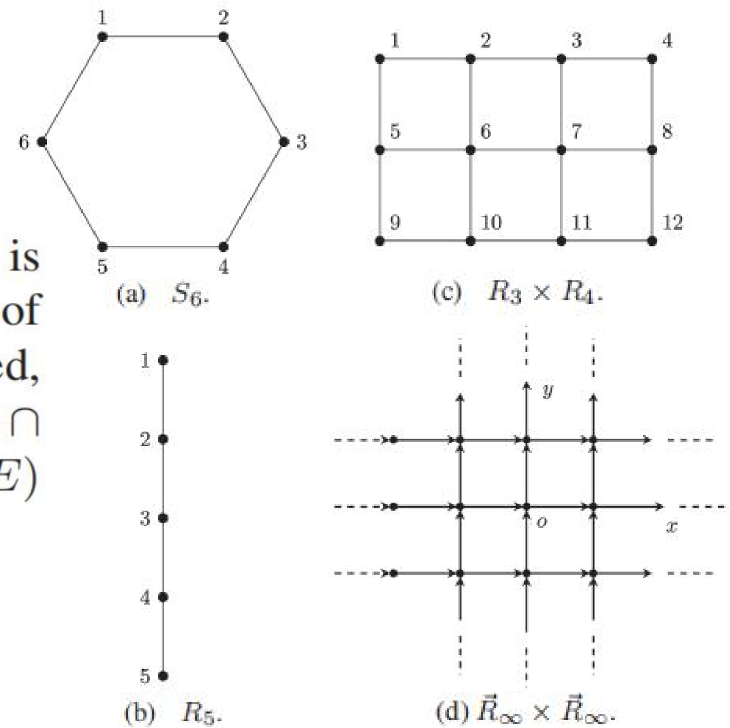

Figure 11: 一些常见的图模型

如果网络图是有向的且所有节点的入度和出度都相同，或网络图是无向的且所有节点的度都相同，则称为**同质网络 (homogeneous network)**；否则，它被称为**异质网络 (heterogeneous network)**。

homogeneous networks: (a),(d) **heterogeneous networks: (b),(c)**

Neighborhood of Node

Definition 1: Let N be the set of nodes in a network, $E \subset N \times N$ the set of edges.

i) $j \in N$ is called a neighbor of i, if either $(i, j) \in E$ or $(j, i) \in E$. Throughout this paper $U(i)$ is used for the set of neighbors of i union $\{i\}$, called the neighborhood of i.

ii) Ignoring the directions of edges, if there exists a path from i to j with length less than or equal to ℓ, then j is said to be an ℓ-neighbor of i, the set of ℓ-neighbors of i is denoted by $U_\ell(i)$. Hence, $U(i) = U_1(i)$, $U_0(i) = \{i\}$.

Fundamental Network Game (FNG)

A normal game consists of **three** factors:

i) n players $N = \{1, 2, \ldots, n\}$;

ii) Player i has the strategy set $S_i = \{1, \ldots, k_i\}$, $i = 1, \ldots, n$, $S := \prod_{i=1}^{n} S_i$ is the set of profiles;

iii) Payoff functions $c_i : S \to \mathbb{R}$, $i = 1, \ldots, n$.

[44] R. Gibbons, A Primer in Game Theory. Glasgow, U.K.: Bell and Bain Ltd., 1992.

Definition 2:

i). A **fundamental game with two players** is called a FNG, if

$$S_1 = S_2 := S_0 = \{1, 2, \ldots, k\}.$$

ii). An FNG is **symmetric**, if

$$c_1(x, y) = c_2(y, x), \qquad \forall\, x, y \in S_0.$$

Strategy Updating Rule (SUR)

Definition 3: An SUR for an NEG, denoted by Π, is a set of mappings

$$x_i(t+1) = g_i(x_j(t), c_j(t); j \in U(i)), \qquad t \geq 0, \quad i \in N.$$

That is, the strategy of each player at time $t+1$ depends on its neighbors' information at t, including their strategies and payoffs.

Note that i) g_i could be a probabilistic mapping, which means a mixed strategy is used by player i; ii) when the network is homogeneous and the SUR used by every player is unique, g_i, $i \in N$, are the same.

Payoff

Definition 4: Let $c_{ij}(t)$ be the payoff of i in the game with j at t. Then the overall payoff of player i at t is

$$c_i(t) = \frac{1}{|U(i)| - 1} \sum_{j \in U(i) \backslash \{i\}} c_{ij}(t), \qquad i \in N \tag{8}$$

where $|U(i)|$ is the cardinality of $U(i)$.

短视最优响应 Mypoic best response asjustment (MBRA)

Construct a set of optimal response set of strategies at t as

$$O_i(t) = \text{argmax}_{s_i \in S_i} c_i(s_i, s^{-i}(t)).$$

此处只考虑
玩家i邻居的
策略

Then

(i) (Case 1) If $x_i(t) \in O_i(t)$, then $x_i(t+1) = x_i(t)$;

(ii) (Case 2) If $x_i(t) \notin O_i(t)$, then

- Deterministic Model (MBRA-D): Choose smallest j, such that $s_j \in O_i(t)$, and set $x_i(t+1) = s_j$.
- Stochastic Model (MBRA-S): Choose any $j \in O_i$, with equal probability $p = 1/|O_i|$.

网络演化博弈的**策略局势动态**由**策略更新规则**决定

无条件模仿 Unconditional Imitation

II-*I* :Unconditional Imitation with Fixed Priority

The best strategy from strategies of neighborhood players $\{j \mid j \in U(i)\}$ at time t is selected as the strategy of player i at time $t+1$, denoted by $x_i(t+1)$. Precisely, if

$$j^* = \text{argmax}_{j \in U(i)} c_j(x(t))$$

then

$$x_i(t+1) = x_{j^*}(t).$$

When the players with the best payoff are not unique, say

$$\text{argmax}_{j \in U(i)} c_j(x(t)) := \{j_1^*, \ldots, j_r^*\}$$

无条件模仿 Unconditional Imitation

II-*II* :Unconditional imitation with equal probability for best strategies. When the best payoff player is unique, it is the same as Π-I. When the players with best payoff are not unique, say, as in (16), then we randomly choose one with equal probability. That is

$$x_i(t+1) = x_{j_\mu^*}(t), \quad \text{with probability} \quad p_\mu^i = \frac{1}{r}$$
$$\mu = 1, \ldots, r.$$

This method leads to a probabilistic k-valued dynamics.

Simplified Fermi Rule

Randomly choose a neighbor $j \in U(i)$, $j \neq i$. Compare $c_j(x(t))$ with $c_i(x(t))$ to determine $x_i(t+1)$ as

$$x_i(t+1) = \begin{cases} x_j(t), & c_j(x(t)) > c_i(x(t)) \\ x_i(t), & \text{otherwise.} \end{cases}$$

This SUR leads to a probabilistic k-valued logical dynamics system.

Definition 5 : An NEG, $((N, E), G, \Pi)$, consists of

i) a network graph (N, E);

ii) an FNG, G, such that if $(i, j) \in E$, then i and j play FNG repetitively with strategies $x_i(t)$ and $x_j(t)$ respectively. Particularly, if the FNG is not symmetric, then the corresponding network must be directed to show that $(i, j) \in E$ implies that in the game i is player one and j is player two;

iii) an SUR, based on local information and expressed as (2).

If the graph is homogeneous, the game is called a homogeneous NEG.

Mathematical Model of NEGs

Modeling of NEGs

Analysis of NEGs

Control of NEGs

[45] D. Cheng, F. He, H. Qi, T. Xu, Modeling, analysis and control of networked evolutionary games, IEEE Transactions on Automatic Control, Vol. 60, No. 9, 2402-2415, 2015.

Modeling of NEGs

Theorem 1: The strategy dynamics of each node can be expressed as

$$x_i(t+1) = f_i(\{x_j(t); j \in U_2(i)\}), \qquad i \in N. \qquad (9)$$

(9) is called the fundamental evolutionary equation (FEE).
We can express (9) into its algebraic form as

$$x_i(t+1) = M_i \ltimes_{j \in U_2(i)} x_j(t), \quad t \geq 0, \; i \in N. \qquad (10)$$

Set $\ell = |U_2(i)|$, then in (10) the $M_i \in \mathcal{L}_{k \times k^\ell}$ when pure strategies are used; and $M_i \in \Upsilon_{k \times k^\ell}$ when mixed strategies are used.

For a homogeneous network all FEEs are the same.

We give an algorithm of FEE as follows

Algorithm 1: Consider a node (player) i.

1) *Step 1:* For each $j \in U(i)$ consider $k \in U(j)$. According to $x_j(t)$ and $x_k(t)$, $c_{j,k}(t)$ can be calculated.
2) *Step 2:* Using formula (9), $c_j(t)$, $j \in U(i)$ can be calculated.
3) *Step 3:* Using the $c_j(t)$, $j \in U(i)$ and according to the SUR, $x_i(t+1)$ can be figured out.

Example 2

We use example to show how to **use the SUR to determine the FEE**. Note that since (10) is a **k-valued logical dynamic system**, it can be expressed into a matrix form (refer to the Appendix).

Example 2: Assume the network is R_3 and the FNG is the game of Rock-Scissors-Paper. The payoff bi-matrix is shown in Table II.

TABLE II
PAYOFF BI-MATRIX (ROCK-SCISSORS-PAPER)

$P_1 \backslash P_2$	$R=1$	$S=2$	$C=3$
$R=1$	$(0,\ 0)$	$(1,\ -1)$	$(-1,\ 1)$
$S=2$	$(-1,\ 1)$	$(0,\ 0)$	$(1,\ -1)$
$C=3$	$(1,\ -1)$	$(-1,\ 1)$	$(0,\ 0)$

Example 2

i) Assume the strategy updating rule is Π–I: If $x_1(t)$, $x_2(t)$, $x_3(t)$ are known, then $x_i(t+1) = f_i(x_1(t), x_2(t), x_3(t))$ can be calculated. For instance, assume $x_1(t) = 1$, $x_2(t) = 2$, $x_3(t) = 3$, then

$$
\begin{aligned}
c_1(t) &= 1, \\
c_{21}(t) &= -1, \quad c_{23}(t) = 1, \Rightarrow c_2(t) = 0. \\
c_3(t) &= -1,
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_1(t+1) &= f_1(x_1(t), x_2(t), x_3(t)) \\
&= x_{\underset{j}{\arg\max}\{c_1(t), c_2(t)\}}(t) = x_1(t) = 1.
\end{aligned}
$$

Example 2

Similarly

$$x_2(t+1) = x_1(t) = 1, \quad x_3(t+1) = x_2(t) = 2.$$

Using the same argument for each profile (x_1, x_2, x_3), f_i, $i = 1, 2, 3$, can be figured out as in Table III.

TABLE III
FROM PAYOFFS TO DYNAMICS

Profile	111	112	113	121	122	123	131	$\cdots$	333
c_1	0	0	0	1	1	1	-1	$\cdots$	0
c_2	0	1/2	-1/2	-1	-1/2	0	1	$\cdots$	0
c_3	0	-1	1	1	0	-1	-1	$\cdots$	0
f_1	1	1	1	1	1	1	3	$\cdots$	3
f_2	1	1	3	1	1	1	3	$\cdots$	3
f_3	1	1	3	1	2	2	3	$\cdots$	3

Example 2

Identifying $i \sim \delta_k^i$, $i = 1, \ldots, k$, we can have the algebraic form of the evolutionary equations as (refer to the Appendix)

$$x_i(t+1) = M_i x(t), \qquad i = 1, 2, 3 \tag{11}$$

where $x_i(t) \in \Delta_3$, $x(t) = \ltimes_{i=1}^3 x_i(t)$, and

$$\begin{aligned}
M_1 &= \delta_3[1\ 1\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 2\ 2\ 2\ 3\ 3\ 3]\\
M_2 &= \delta_3[1\ 1\ 3\ 1\ 1\ 1\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 1\ 2\ 2\ 3\ 2\ 3]\\
M_3 &= \delta_3[1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3].
\end{aligned}$$

Consider an NEG and assume $S_0 = \{1, \cdots, k\}$. Identifying $i \sim \delta_k^i$, then for each i, there exists matrix $M_i \in \mathcal{M}_{k \times k^{l_i}}$ (in probabilistic case, $M_i \in \Upsilon_{k \times l_i}$), $l_i = |U_2(i)|$, such that

$$x_i(t+1) = M_i \ltimes_{j \in U_2(i)} x_j(t), \quad i \in N,$$

where M_i is the structure matrix of f_i.

Assume $X \in \Upsilon_p$ and $Y \in \Upsilon_q$. Define two matrices $D_r^{q,p} = I_p \otimes \mathbf{1}_q^T$, $D_f^{p,q} = \mathbf{1}_p^T \otimes I_q$, respectively. Then we have

$$D_r^{p,q} XY = X; \; D_f^{p,q} XY = Y.$$

Thus, we assume the strategy dynamics has its algebraic form

$$x_i(t+1) = M_i x(t), \; i = 1, \cdots, n.$$

Multiplying all the equations together, we have **the algebraic state space** form of strategy dynamics as follows:

$$x(t+1) = Mx(t),$$

where $M = M_1 * M_2 * \cdots * M_n$.

For more details about Khatri-Rao product (∗) , see Appendix

Example 2

We can get the game **transition matrix** immediately as

$$
\begin{aligned}
M_G &= M_1 * M_2 * M_3 \\
&= \delta_{27}[1,1,9,1,2,2,27,23,27,1,1,9,10,14,14,15,14, \\
&\qquad 15,25,25,29,10,14,14,27,23,27].
\end{aligned}
$$

Since

$$
\begin{aligned}
(M_G)^\ell = \delta_{27}[&1,1,27,1,1,1,27,14,27,1,1,27,1,14,14,14, \\
&14,14,27,27,27,1,14,14,27,14,27], \qquad \ell \geq 2
\end{aligned}
$$

we can figure out

a) there are three fixed points: $\delta_{27}^{1} \sim (1,1,1)$, $\delta_{27}^{14} \sim (2,2,2)$, $\delta_{27}^{27} \sim (3,3,3)$;

b) the **corresponding basins** (i.e., regions of attraction) of these **three attractions** (fixed points) are, respectively.

$$B_1 = \delta_{27}\{1,2,4,5,6,10,11,13,22\}$$
$$B_2 = \delta_{27}\{8,14,15,16,17,18,23,24,26\}$$
$$B_3 = \delta_{27}\{3,7,9,12,19,20,21,25,27\}$$

c) there is **no cycle**.

So the network **converges to** one of **three fixed points** with **equal probability** (as the initial strategy is uniformly distributed).

In **homogeneous case** (with unique SUR) the NEG dynamics is determined by the **unique FEE**.

First, we consider how to **calculate the SPD using the unique FEE**.

Example 3

Example 3: Consider an NEG $((N,E),G,\Pi)$, where $(N,E)=S_5$; G is the Prisoner's Dilemma defined in Example 2 with parameters $R=-1$, $S=-10$, $T=0$, $P=-5$; and the strategy updating rule Π–I is chosen. (In fact, in this case Π–I and Π–II lead to the same dynamics.)

We first calculate FEE (9) for an arbitrary node i. Note that on S_n the neighborhoods of i are $U(i)=\{i-1,i,i+1\}$, $U_2(i)=\{i-2,i-1,i,i+1,i+2\}$, hence (9) becomes

$$x_i(t+1)=f(x_{i-2}(t),x_{i-1}(t),x_i(t),x_{i+1}(t),x_{i+2}(t)), \quad i=1,2,$$

Example 3

Using the **swap matrix**, it is easy to see that

$$
\begin{aligned}
x_1(t+1) &= L_5 x_4(t) x_5(t) x_1(t) x_2(t) x_3(t) = L_5 W_{[2^3,2^2]} x(t) \\
x_2(t+1) &= L_5 x_5(t) x_1(t) x_2(t) x_3(t) x_4(t) = L_5 W_{[2^4,2]} x(t) \\
x_3(t+1) &= L_5 x_1(t) x_2(t) x_3(t) x_4(t) x_5(t) = L_5 x(t) \\
x_4(t+1) &= L_5 x_2(t) x_3(t) x_4(t) x_5(t) x_1(t) = L_5 W_{[2,2^4]} x(t) \\
x_5(t+1) &= L_5 x_3(t) x_4(t) x_5(t) x_1(t) x_2(t) = L_5 W_{[2^2,2^3]} x(t)
\end{aligned}
$$

where $x(t) = \ltimes_{j=1}^{5} x_j(t)$.

Finally, we have the **evolutionary dynamic equation** as

$$x(t+1) = M_5 x(t)$$

$$
\begin{aligned}
M_5 = & \left(L_5 W_{[2^3,2^2]}\right) * \left(L_5 W_{[2^4,2]}\right) * L_5 \\
& * \left(L_5 W_{[2,2^4]}\right) * \left(L_5 W_{[2^2,2^3]}\right) \\
= & \delta_{32}[1\ 20\ 8\ 4\ 15\ 32\ 7\ 32\ 29\ 32\ 32\ 32\ 13\ 32\ 32\ 32\ 26\ 18 \\
& 32\ 32\ 32\ 32\ 32\ 32\ 25\ 32\ 32\ 32\ 32\ 32\ 32\ 32].
\end{aligned}
$$

The FEE can be used to calculate not only the **strategy evolutionary equation** for S_5 , but also for any $S_n, n > 2$.

Then the **evolutionary dynamic properties** can be found via the corresponding **transition matrix**. We are more interested in the case of **large** n.

For x_i, we have

$$\begin{aligned} x_1(t+1) &= L_5 x_{n-1}(t) x_n(t) x_1(t) x_2(t) x_3(t) \\ &= L_5 D_r^{2^5,2^{n-5}} x_{n-1}(t) x_n(t) x_1(t) x_2(t) \cdots x_{n-2}(t) \\ &= L_5 D_r^{2^5,2^{n-5}} W_{[2^{n-2},2^2]} x_1(t) \cdots x_n(t) \\ &:= H_1 x(t), \end{aligned}$$

where L_5 is the structure matrix of f.

Similarly, we obtain a general expression as follows:

$$x_i(t+1) = H_i x(t),\ \ i = 1, \cdots, n,$$

where $H_i = L_5 D_r^{2^5, 2^{n-5}} W_{[2^{\alpha(i)}, 2^{n-\alpha(i)}]}$, $i = 1, \cdots, n$ and $\alpha(i) = \begin{cases} i-3, & i \geq 3; \\ i-3+n, & i < 3. \end{cases}$

Finally, the **profile transition matrix** can be calculated by

$$M_n = H_1 * H_2 * \cdots * H_n.$$

Now, we give **an algorithm** to describe how to **calculate the SPDs using FEE**.

Algorithm 2

1) *Step 1:* From the FEE (9) to calculate its algebraic form

$$x_i(t+1) = M_i \ltimes_{j \in U_2(i)} x_j(t), \quad i = 1, \ldots, n$$

where $M_i \in \mathcal{L}_{k \times k^{|U_2(i)|}}$.

2) *Step 2:*

$$x_i(t+1) = W_i \ltimes_{j=1}^{n} x_j, \quad i = 1, \ldots, n.$$

W_i is derived by adding some dummy factors which make the product in step 1 can be a product of all factors.

3) *Step 3:* Denote by $x := \ltimes_{j=1}^{n} x_j$. The SPDs can be constructed as

$$x(t+1) = Lx(t)$$

where $L \in \mathcal{L}_{k^n \times k^n}$ is determined by

$$L = W_1 * W_2 * \ldots * W_n.$$

The algebraic form of the SPDs is the dynamics of the NEG

Evolutionarily Stable Strategy (ESS)

J. M. Smith 和 G. R. Price 提出了**演化稳定策略**的基本概念，该**均衡概念**的提出使得演化博弈理论的有有了明确的方向，为进化博弈论的进一步发展奠定了坚实的基础。

To answer this question, we need a more precise definition of an ESS. We define $E_J(I)$ as the expected pay-off to I played against J. Then I is an ESS if, for all J, $E_I(I) > E_I(J)$; if for any strategy J, $E_I(I) = E_I(J)$, then evolutionary stability requires that $E_J(I) > E_J(J)$. The relevance of the latter condition is as follows. If in a population adopting strategy I a mutant J arises whose expectation against I is the same as I's expectation against itself, then J will increase by genetic drift until meetings between two J's becomes a common event.

[1] J. M. Smith, G. R. Price, The logic of animal conflict, Nature, Vol. 246, No. 5427, 15-18, 1973.

ESS of NEGs

The ESS is a fundamental concept for evolutionary games. It is natural to extend it to the NEGs. Hence, we need **a new precise definition** of the **ESS** for **NEGs**.

Definition 6 :

1) For a given NEG a strategy $\xi \in S$ is called an ESS, if there exists a $\mu \geq 1$, such that as long as the initial strategy profile y_0 satisfies

$$\|y_0 - x_0\| \leq \mu$$

we have

$$\lim_{t \to \infty} y(t, y_0) = x_0$$

where $x_0 = \xi^n$. Moreover, ξ is called the ESS of level μ.

ESS of NEGs

- When the population n is **finite**, there exists a $T > 0$ such that

$$y(t, y_0) = x_0, \quad t \geq T.$$

- It is clear that the μ can be used to measure the robustness of the stability. So **the higher the level the more robust the ESS**.

Example 4

Consider a NEG with following SPD:

$$x(t+1) = Lx(t)$$

where $x(t) = \ltimes_{i=1}^{7} x_i(t)$, and

$$L = \delta_{128}$$
$$\begin{array}{llllllllllll}
[1 & 68 & 8 & 72 & 15 & 80 & 16 & 80 & 29 & 96 & 32 & 96 \\
31 & 96 & 32 & 96 & 57 & 124 & 64 & 128 & 63 & 128 & 64 & 128 \\
61 & 128 & 64 & 128 & 63 & 128 & 64 & 128 & 113 & 116 & 120 & 120 \\
127 & 128 & 128 & 128 & 125 & 128 & 128 & 128 & 127 & 128 & 128 & 128 \\
121 & 124 & 128 & 128 & 127 & 128 & 128 & 128 & 125 & 128 & 128 & 128 \\
127 & 128 & 128 & 128 & 98 & 100 & 104 & 104 & 112 & 112 & 112 & 112 \\
126 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 122 & 124 & 128 & 128 \\
128 & 128 & 128 & 128 & 126 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
114 & 116 & 120 & 120 & 128 & 128 & 128 & 128 & 126 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 122 & 124 & 128 & 128 & 128 & 128 & 128 & 128 \\
126 & 128 & 128 & 128 & 128 & 128 & 128 & 128]. & & & &
\end{array}$$

It is easy to calculate that

$$
\begin{aligned}
L^k = \delta_{128}\, & [1\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128 \\
& 128\ 128\ 128\ 128\ 128\ 128\ 128\ 128]
\end{aligned}
$$

where $k \geq 3$.

It is clear that unless $x(0) = \delta_{128}^1$, which leads to $x(\infty) = x(3) = \delta_{128}^1 \sim (1, 1, 1, 1, 1, 1, 1)$, any other initial states converge to $\delta_{128}^{128} \sim (2, 2, 2, 2, 2, 2, 2)$.

We conclude that $\xi = \delta_2^2 \sim 2$ (i.e., strategy 2) is an ESS. In addition, it is so strong that we can choose $\mu = 6$, and as long as $|y_0 - x_0| \leq \mu$, (where $x_0 = \xi^7$), and setting $T = 3$, (36) holds. Hence, the ESS is of level 6.

Control of NEGs

Definition 7: Let $((N,E),G,\Pi)$ be an NEG, $\{X,U\}$ be a partition of N, i.e., $X \cap U = \emptyset$ and $N = X \cup U$. Then $((X \cup U, E), G, \Pi)$ is called a control NEG, if the strategies for nodes in U, denoted by $u_j \in U$, $j = 1, \dots, |U|$, can be assigned at each moment $t \geq 0$. Moreover, $x \in X$ is called a state and $u \in U$ is called a control.

Definition 8:

1) A state x_d is said to be $T(>0)$ step reachable from $x(0)=x_0$, if there exists a sequence of controls $u_0,\dots,$ u_{T-1} such that $x(T)=x_d$. The set of T step reachable states is denoted as $R_T(x_0)$;
2) The reachable set from x_0 is defined as

$$R(x_0) := \cup_{T=1}^{\infty} R_T(x_0).$$

3) A state x_e is said to be stabilizable from x_0, if there exists a control sequence $u_0, u_1, \dots$ and a $T(>0)$, such that the trajectory from x_0 converges to x_e, precisely, $x(t)=x_e$, $t \geq T$. x_e is stabilizable, if it is stabilizable from any $x_0 \in \mathcal{D}_k^n$.

Next, we consider the **dynamics of a control NEG**

For each $u \in \Delta_{k^m}$, we can have a control-dependent profile transition matrix, defined as

$$M\left(u=\delta_{k^m}^i\right) := M_i, \qquad i=1,2,\dots,k^m.$$

Define the set of **control-dependent strategy transition matrices** by

$$\mathcal{M}_U = \{M_1, \cdots, M_{k^m}\}.$$

Proposition 1: Consider a control NEG $((X \cup U, E), G, \Pi)$, with $|X| = n$, $|U| = m$, $|S_0| = k$.

1) x_d is reachable from x_0, if and only if there exists a sequence $\{M_{i_0}, M_{i_1}, \ldots, M_{i_{T-1}}\} \subset \mathcal{M}_U$, $T \leq k^n$, such that

$$x_d = M_{i_{T-1}} M_{i_{T-2}} \ldots M_{i_1} M_{i_0} x_0.$$

2) x_d is stabilizable from x_0, if and only if i) x_d is reachable from x_0 and there exists at least one $M^* \in \mathcal{M}_U$, such that x_d is a fixed point of M^*.

Next, we consider the **consensus of control NEGs**.

Definition 9: Let $\xi \in \Delta_k$. An NEG with $|N| = n$ is said to reach a consensus at ξ if it is stabilizable to $x_e = \xi^n$.

Proposition2 :

1) An NEG cannot reach a consensus, if there are more than one common fixed point for all $M \in \mathcal{M}_U$.
2) An NEG can reach a consensus ξ, the NEG is stabilizable to $x_e = \xi^n$.

Example 4

i) Consider the Prisoner's Dilemma Game over S_6 with strategy updating rule $\Pi = \Pi–I$, where $\{x_1, x_2, x_3, x_4, x_5\}$ are normal players, called the states, and $x_6 = u$ is the control, connected to x_1 and x_5.

The control-dependent strategy transition matrices are

$$M(u = \delta_2^1) = M_1$$
$$= \delta_{32}[17, 2, 24, 4, 31, 32, 32, 32, 29, 32, 32, 32, 31, 32, 32, 32, 17, 18, 32, 32, 32, 32, 32, 32, 25, 32, 32, 32, 32, 32, 32, 32]$$
$$M(u = \delta_2^2) = M_2$$
$$= \delta_{32}[1, 4, 8, 4, 15, 16, 7, 8, 29, 32, 32, 32, 13, 32, 15, 32, 26, 28, 32, 32, 32, 32, 32, 32, 26, 32, 32, 32, 30, 32, 32, 32]. \quad (51)$$

We can see that there are two common fixed points: $x_e^1 = \delta_{32}^4$ and $x_e^2 = \delta_{32}^{32}$. Hence the NEG cannot reach consensus.

ii) Next, we add another control, so that, x_1, x_2, x_3, x_4, x_5, u_1, u_2 form an S_7. Then the control-dependent strategy transition matrices become

$$
\begin{aligned}
&M(u_1=\delta_2^1, u_2=\delta_2^1)=M_1\\
&=\delta_{32}[1,1,8,4,16,16,8,8,29,29,32,32,13,16,16,16,28,25,\\
&\qquad 32,32,32,32,32,32,28,32,32,32,32,32,32,32]\\
&M(u_1=\delta_2^1, u_2=\delta_2^2)=M_2\\
&=\delta_{32}[1,1,8,4,16,16,16,16,29,29,32,32,29,32,32,32,20,\\
&\qquad 17,32,32,32,32,32,32,28,32,32,32,32,32,32,32]\\
&M(u_1=\delta_2^2, u_2=\delta_2^1)=M_3\\
&=\delta_{32}[17,17,24,20,32,32,24,24,29,29,32,32,29,32,32,32,\\
&\qquad 28,25,32,32,32,32,32,32,28,32,32,32,32,32,32,32]
\end{aligned}
$$

$$
\begin{aligned}
& M(u_1 = \delta_2^2, u_2 = \delta_2^2) = M_4 \\
= & \delta_{32}[1, 1, 8, 4, 32, 32, 32, 32, 29, 29, 32, 32, 29, 32, 32, 32, 20, \\
& \quad 17, 32, 32, 32, 32, 32, 32, 28, 32, 32, 32, 32, 32, 32, 32].
\end{aligned}
$$

It is ready to check that there is a common fixed point: $x_e = \delta_{32}^{32}$, where $x_e = \xi^5$ with $\xi = \delta_2^2$. Moreover, x_e is reachable from any $x(0)$. Therefore, the NEG can reach consensus at $x_e = \xi^5$, where $\xi = \delta_2^2$.

NEGs based on STP

近年来，很多专家学者将STP应用到演化博弈论的研究中，取得了一系列成果：

- 文献[46]为NEGs提供了一个严格的**数学模型**。利用基本演化方程，博弈中策略组合升级过程被表示为一个k值(确定的或者概率的)**逻辑动态网络**，基于此来分析**网络动态行为**。
- 文献[47]提出了NEGs的**ESS的定义**，并说明了和传统演化稳定策略定义的一致性。
- 此外，还有对网络博弈混合策略**纳什均衡点**[48]、**受输入扰动**的博弈**最优控制**[49]、**超网络势演化**博弈动态[50]、**时滞**网络演化博弈[51]等的研究。

[46] D. Cheng, F. He, H. Qi, T. Xu, Modeling, analysis and control of networked evolutionary games, IEEE Transactions on Automatic Control, Vol. 60, No. 9, 2402-2415, 2015.
[47] D. Cheng, T. Xu, H. Qi. Evolutionarily stable strategy of networked evolutionary games, IEEE Transactions on Neural Networks and Learning Systems, Vol. 25, No. 7, 1335-1345, 2014.
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[49] M. Yang, T. Chu, Optimal control of logical control network with noisy inputs, Proceedings of the American Control Conference, 2848-2853, 2012.
[50] T. Liu, Y. Wang, D. Cheng, Dynamics and stability of potential hyper.networked evolutionary games, Proceedings of the 34th Chinese Control Conference, 9090-9097, 2015.
[51] Y. Wang, D. Cheng. Dynamics and stability of evolutionary games with timeinvariant delay in strategies, Proceedings of the 27th Chinese Control and Decision Conference, 6427-6432, 2015.

基于STP方法，目前博弈论方向已经有了一些初步的结果

Stability and stabilization

下述文献研究了**NEGs**的稳定和镇定等相关问题

[52] D.Cheng, T. Xu, H. Qi, Evolutionarily **stable** strategy of networked evolutionary games, IEEE Transactions on Neural Networks & Learning Systems, Vol. 25, No. 7, 1335-1345, 2014.

[53] X. Ding, H. Li, Q. Yang, Y. Zhou, A. Alsaedi, F. E. Alsaadi, **Stochastic stability and stabilization** of n-person random evolutionary Boolean games, Applied Mathematics & Computation, No. 306, 1-12, 2017.

[54] T. Liu, Y. Wang, D. Cheng, **Dynamics and stability** of potential hyper-networked evolutionary games, Int. J. Aut. Comput., Vol. 14, No.2, 229-238, 2017.

[55] Y. Wang , D. Cheng , **Dynamics and stability** for a class of evolutionary games with time delays in strategies. Science China Information. Science, Vol. 59, No.9, 092209, 2016.

[56] Y. Wang, D. Cheng, **Stability and stabilization** of a class of finite evolutionary games, Journal of the Franklin Institute, Vol. 354, No. 3, 1603-1617, 2017.

[57] H. Li, X. Ding, A. Alsaedi, F.E. Alsaadi, **Stochastic set stabilization** of n-person random evolutionary Boolean games and its applications, Applied Mathematics & Computation, Vol. 11, No. 13, 2152-2160. 2017.

下述文献研究了**时滞**NEGs等相关问题

Time delay

[58] G. Zhao, Y. Wang, H. Li, A matrix approach to the modeling and analysis of networked evolutionary games with **time delays**, IEEE/CAA Journal of Automatica Sinica, Vol. 5, No. 4, 818-826, 2018.

[59] Y. Zheng, C. Li, J. Feng, Modeling and dynamics of networked evolutionary game with **switched time delay**, IEEE Transactions on Control of Network Systems, Doi: 10.1109/TCNS.2021.3084548, 2021.

[60] J. Wang, R. Leone, S. Fu, J. Xia, L. Qiao, Event-triggered control design for networked evolutionary games with **time invariant delay** in strategies, International Journal of Systems Science, Doi: 10.1080/00207721. 2020.1831646, 2020.

[61] S. Fu, G. Zhao, H. Li, A. Alsaedi, F. E. Alsaadi, Model and control for a class of networked evolutionary games with finite memories and **time varying networks**, Circuits, Systems, and Signal Processing, Vol. 37, No. 7, 3093-3114, 2018.

[62] S. Fu, H. Li, G. Zhao, Modeling and strategy optimization for a kind of networked evolutionary games **with memories** under the bankruptcy mechanism, International Journal of Control, Vol. 91, No. 5, 1104-1117, 2018.

Consensus

[63] Y. Li, H. Li, X. Xu, Y. Li, 2018. Semi-tensor product approach to minimal-agent **consensus** control of networked evolutionary games, IET Contr. Theor. Appl., Vol. 12, No. 16, 2269-2275, 2018.

[64] G. Zhao, H. Li, W. Sun, et al, Modelling and strategy **consensus** for a class of networked evolutionary games. Int. J. Sys. Sci., Vol. 49, No. 12, 2548-2557, 2018.

Optimisation

[65] S. Fu, Y. Pan, J. Feng, J. Zhao, Strategy **optimisation** for coupled evolutionary public good games with threshold, Int. J. Contr., doi: 10.1080/00207179.2020.1803411, 2020.

[66] P. Guo, Y. Wang, H. Li, Algebraic formulation and strategy **optimization** for a class of evolutionary networked games via semi-tensor product method, Automatica, Vol. 49, 3384-3389, 2013.

(1). Limitation on SUR

Unfortunately, some useful SURs can not be included in this class. For instance, the FP (fictitious play), which needs all the historic knowledge to update its strategy; the SAP (spatial adaptive player) which has time-varying topology .

Roughly speaking, most learning SURs cannot be formulated by (9) directly, **which are left for further study**.

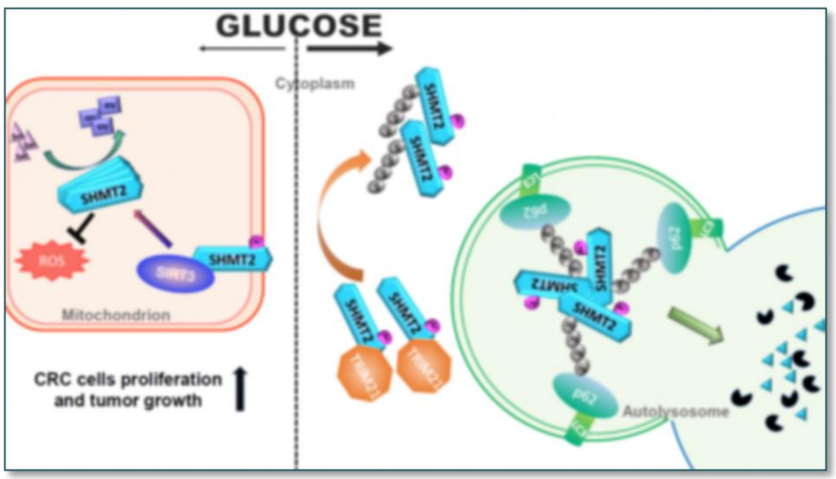
(2). Computational Intractability

If we want to distinct NEGs with different network topologies precisely but not statistically, the complexity is intrinsic.

It was pointed out in [67]: "The main challenge that faced in studying strategic interaction in social settings is the **inherent complexity of networks**. Without focusing in on specific structures in terms of the games, it is hard to draw any conclusions."

[67] M. O. Jackson, Y. Zenou, Games on Networks, Handbook of Game Theory with Economic Applications, No. 4, 95-163, 2015.

Colitis-associated colon cancer (CACC) network



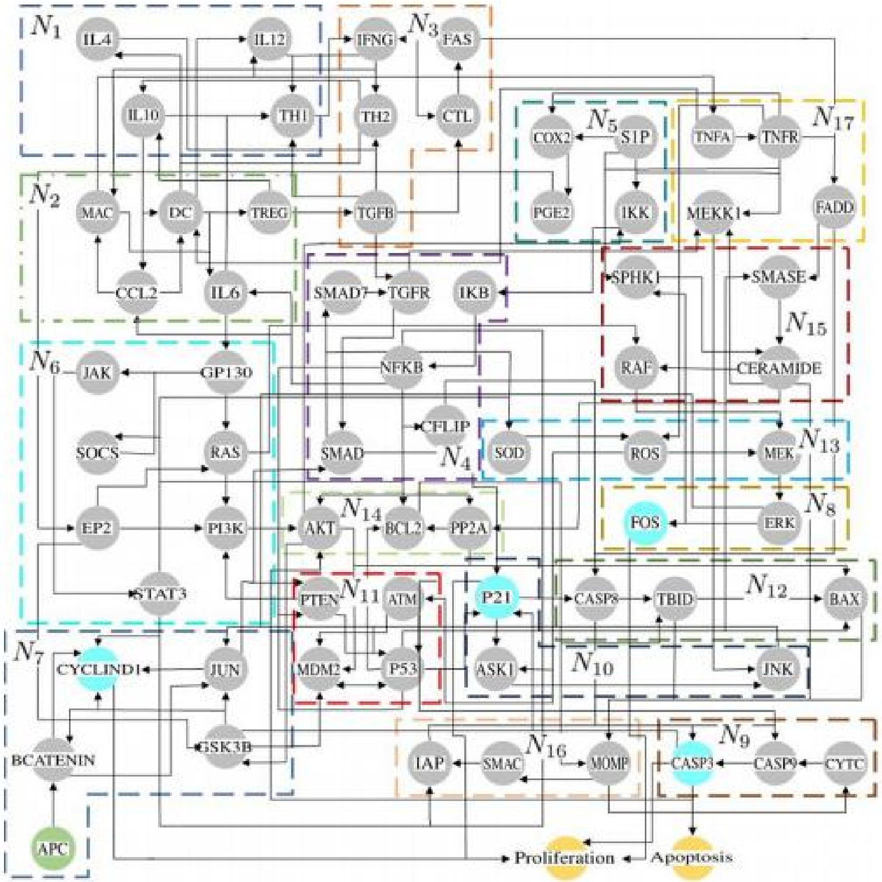


Figure 12: Colitis-associated colon cancer (CACC) 网络模型

Based on the STP method, many efficient techniques have been introduced to solve the control problems of large-scale logical control networks, including **approximation method** [68], **network aggregation approach** [69]–[71], **logical matrix factorization technique** [72], and **pinning control design method** [73].

[68] D. Cheng, Y. Zhao, J. Kim, Y. Zhao, Approximation of Boolean networks, Proceedings of the 10th World Congress on Intelligent Control and Automation, 2012, pp. 2280–2285.
[69] K. Zhang, K. H. Johansson, Efficient verification of observability and reconstructibility for large Boolean control networks with special structures, IEEE Transactions on Automatic Control, Vol. 65, No. 12, 5144-5158, 2020.
[70] Y. Zhao, J. Kim, M. Filippone, Aggregation algorithm towards large-scale Boolean network analysis, IEEE Transactions on Automatic Control, Vol. 58, No. 8, 1976–1985, 2013.
[71] Y. Zhao, B. K. Ghosh, D. Cheng, Control of large-scale Boolean networks via network aggregation, IEEE Transactions on Neural Networks and Learning Systems, Vol. 27, No. 7, 1527-1536, 2016.
[72] H. Li and Y. Wang, Logical matrix factorization with application to topological structure analysis of Boolean network, IEEE Transactions on Automatic Control, Vol. 60, No. 5, 1380–1385, 2015.
[73] J. Zhong, D. W. C. Ho, J. Lu, A new approach to pinning control of Boolean networks, arXiv:1912.01411. [Online]. Available:http://arxiv.org/abs/1912.01411, 2019.
[74] Y. Yu, J.-E. Feng, J. Pan, D. Cheng, Block decoupling of Boolean control networks, IEEE Transactions on Automatic Control, Vol. 64, No. 8, 3129-3140, 2019.

The approximation of logical networks was proposed by Cheng and Zhao[68] to obtain a **simplified network** of **large-scale logical networks**.

A. Approximation method

[70] firstly introduced the network aggregation approach for the attractors analysis of large-scale logical networks.

B. **Network aggregation method**

Network aggregation method

Consider the following Boolean network:

$$
\begin{aligned}
x_1(t+1) &= f_1[x_1(t), x_2(t), \ldots, x_n(t)] \\
x_2(t+1) &= f_2[x_1(t), x_2(t), \ldots, x_n(t)] \\
&\cdots \\
x_n(t+1) &= f_n[x_1(t), x_2(t), \ldots, x_n(t)]
\end{aligned}
$$

where $x_i(t)$ for $i = 1, 2, \ldots, n$ denotes the state of node x_i at time t that can be either 0 for inactive or 1 for active. The nodes

The nodes can be **partitioned into s-number** of blocks as follows:

$$\mathcal{X} = \{x_1, x_2, \ldots, x_n\} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_s$$

where $\mathcal{X}_i$ is a proper subset of $\mathcal{X}$, $\mathcal{X}_i \cap \mathcal{X}_j$ is empty for $i \neq j$, $\mathcal{X}_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$, n_i is the number of nodes in the ith block, and x_{ij}, the jth node in the ith block, is equal to x_k for a $k \in \{1, 2, \ldots, n\}$.

We call this partition an aggregation of Boolean network!

Each block $\mathcal{X}_i$ has **incoming edges** from outside of the block and some **outgoing edges** to the outside. The source nodes of these edges can be interpreted as **inputs** and **outputs** for each block. Denote the set of inputs and outputs of the block $\mathcal{X}_i$ as

$$\mathcal{U}_i = \{u_{i1}, u_{i2}, \ldots, u_{im_i}\} \text{ and } \mathcal{Y}_i = \{y_{i1}, y_{i2}, \ldots, y_{ip_i}\},$$

The set of all source nodes, whose edges cut by the **partition**, as

$$\mathcal{C} = \{x_{c_1}, x_{c_2}, \ldots, x_{c_n}\}.$$

Then, the subnetwork Σ_i, with nodes in $\mathcal{X}_i$ and inputs in $\mathcal{U}_i$, is a Boolean control network given by

$$\Sigma_i : x_{ij}(t+1) = f_{ij}[x_{i1}(t), x_{i2}(t), \ldots, x_{in_i}(t), u_{i1}(t), u_{i2}(t), \ldots, u_{im_i}(t)]$$

for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, n_i$.

- 1) **Attractors** analysis of large-scale logical networks[70]
- 2) **Controllability** analysis of large-scale logical networks[71]
- 3) **Observability** analysis of large-scale logical networks[69]
- 4) **Stabilization** of large-scale logical networks[71]

Example 5

Consider a Boolean network example in Figure 13. Assume its dynamics is described as

$$
\begin{cases}
x_1(t+1) = x_2(t) \\
x_2(t+1) = x_3(t) \wedge x_7(t) \\
x_3(t+1) = x_1(t) \leftrightarrow x_2(t) \\
x_4(t+1) = (x_1(t) \vee x_5(t)) \rightarrow x_7(t) \\
x_5(t+1) = \neg x_4(t) \\
x_6(t+1) = x_6(t) \bar{\vee} x_8(t) \\
x_7(t+1) = x_6(t) \\
x_8(t+1) = x_7(t) \vee x_9(t) \\
x_9(t+1) = \neg x_6(t)
\end{cases}
$$

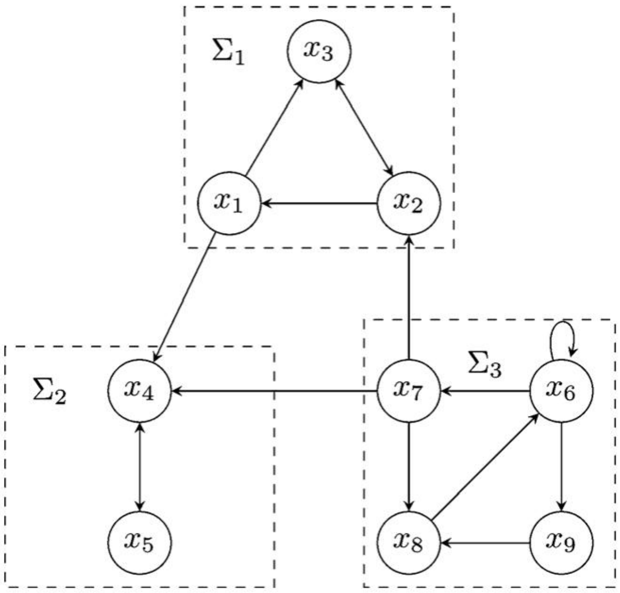


Figure 13: Example of aggregation of a network comprising nine nodes into three Boolean control networks.

where $\leftrightarrow$, $\rightarrow$, and $\bar{\vee}$ denote "EQUIVALENCE," "IMPLICATION," and "EXCLUSIVE-OR" operations respectively. Consider the aggregation into 3 blocks

$$\{x_1, x_2, x_3\} \in \mathcal{X}_1,\ \{x_4, x_5\} \in \mathcal{X}_2,\ \{x_6, x_7, x_8, x_9\} \in \mathcal{X}_3.$$

Now the **inputs** and **outputs** of **each subsystem** are

$$\mathcal{U}_1 = \{u_{11} = x_7\},\ \mathcal{U}_2 = \{u_{21} = x_1, u_{22} = x_7\},\ \mathcal{U}_3 = \emptyset$$

$$\mathcal{Y}_1 = \{y_{11} = x_1\},\ \mathcal{Y}_2 = \emptyset,\ \mathcal{Y}_3 = \{y_{31} = x_7\}$$

$$\mathcal{C} = \{x_{c_1} = x_1, x_{c_2} = x_7\}.$$

Hence, there are three subnetworks:

$$\Sigma_1 : \begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = x_3(t) \wedge u_{11}(t) \\ x_3(t+1) = x_1(t) \leftrightarrow x_2(t); \end{cases}$$

$$\Sigma_2 : \begin{cases} x_4(t+1) = (u_{21}(t) \vee x_5(t)) \rightarrow u_{22}(t) \\ x_5(t+1) = \neg x_4(t); \end{cases}$$

$$\Sigma_3 : \begin{cases} x_6(t+1) = x_6(t) \bar{\vee} x_8(t) \\ x_7(t+1) = x_6(t) \\ x_8(t+1) = x_7(t) \vee x_9(t) \\ x_9(t+1) = \neg x_6(t). \end{cases}$$

Note that the aggregation shown in Example 5 is **not unique** but there are many other different configurations.

Aggregation Method to Large-Size NEMGs

It is a meaningful attempt to investigate the strategy consensus analysis and synthesis of **large-size networked evolutionary matrix game (NEMGs)** with arbitrary network structure by virtue of aggregation method.

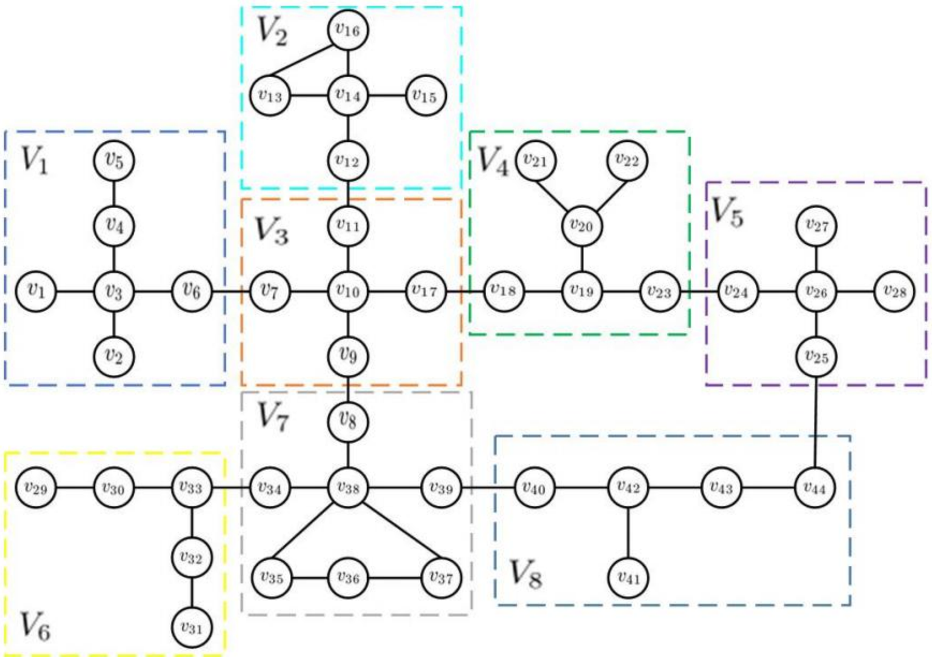


Figure 14: Network graph of the NEMG

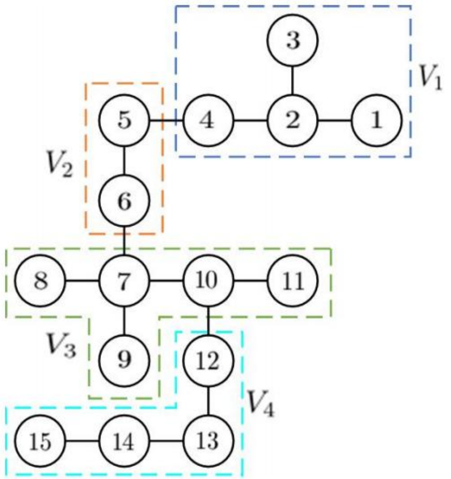
Illustrative example

Consider the NEMG of international trade between several countries, where the network graph is given in Fig. 4. The nodes $1, 2, \cdots, 15$, denote some countries including China, Japan, Russia, USA, etc. According to [12], the trade between two countries is generally depicted by the dominant strategic equilibrium of "Cooperation-Cooperation". Denote the strategies "Cooperation" and "Defection" by "1" and "2", respectively. For the ease of computation, we consider the payoff matrix with

$$\pi^{i,j} = \pi^{j,i} := \begin{bmatrix} 12 & 6 \\ 8 & 7 \end{bmatrix}, \ (i,j) \in E.$$

In addition, the SUR considered in this example is Unconditional Imitation with Fixed Priority [4], where the payoff of each player is given in (1).

$$p_i(t) := \sum_{j \in N(i) \setminus \{i\}} [\pi^{i,j}]_{v_i(t), v_j(t)},$$



Partial international trade network

To sum up, by Theorem 4, under the control strategy $u_j(t) \equiv 1, j = 1, 2$, the considered control NEMG reaches a strategy consensus at strategy "Cooperation" (see Fig. 5).

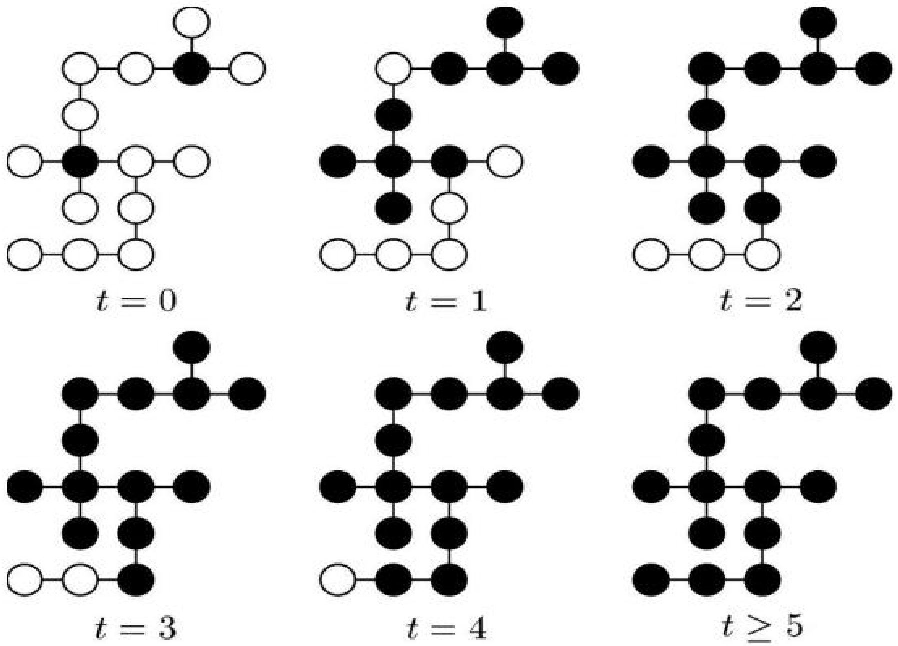


Fig. 5: Evolution of strategy for each country in partial international trade network, where the solid dot represents "Cooperation".

ESS of General NEGs

Now, we considers the general case, where the network is, in general, **heterogeneous**. We need the following assumption.

A1: There exist two numbers p and q satisfying $1 \le p \le q < \infty$, such that

$$p \le \text{degree}(i) \le q, \quad \forall i \in N.$$

When the network size is **small**, the ESS can be verified via its SPDs. To deal with **the networks of large size**, now, we propose a method called the **decomposition approach**.

Consider a general NEG and assume A1. Let $O \in N$ be any node. If there exist two integers $\mu \geq 1$ and $r \geq 1$, such that any $x_j(t_0)$, $j \in U_{2r}(O)$, which is the initial strategy profile of $\{j \mid j \in U_{2r}(O)\}$ with $\sum_{j \in U_{2r}(O)} |x_j(t_0) - \xi| \leq \mu$, satisfying

$$x_O^i(t_0 + \ell) = x_O^j(t_0 + \ell) \quad \forall i, j \in U(O) \backslash \{O\} \ell = 1, \ldots, r-1$$

$$x_O^i(t_0 + r) = x_O^j(t_0 + r) = \xi \quad \forall i, j \in U(O) \backslash \{O\}$$

then ξ is an ESS of level $\mu/[2r]$.

If in **every branch** O converges to ξ , then in the **overall NEG** converges to ξ too.

Stationary Stable Profiles of NEGs

Definition 10: Consider an NEG. Assume there exists a $T > 0$ such that the profile is eventually constant as

$$x_i(t) := p_i, \qquad t \geq T; \quad i = 1, \ldots, n.$$

Then $\{p_i \mid i = 1, \ldots, n\}$, or equivalently, $p = \ltimes_{i=1}^{n} p_i$, is called the stationary stable profile, and the smallest $T(> 0)$ is the reaching time.

The concept of **stationary stable profiles** is presented !

We consider how to find the **stationary stable profiles** for **large-scale homogeneous NEGs**.

Theorem 2: An NEG has a stationary stable profile, if and only if there exists an $\ell > 0$ such that

$$\left[M_{U_{2(\ell+1)}}\right]^{\ell+1} = \left[M_{U_{2(\ell+1)}}\right]^{\ell}.$$

Moreover, let T be the smallest such ℓ, called the reaching time. Then the stationary stable profile at θ is

$$p_\theta = \theta(\infty) = \pi_\beta \left[M_{U_{2T}}\right]^T \ltimes_{i \in U_{2T}} \bar{x}_i(0).$$

[75] D. Cheng, F. He, H. Qi, T. Xu, Modeling, analysis and control of networked evolutionary games, IEEE Transactions on Automatic Control, Vol. 60, No. 9, 2402-2415, 2015.

- I. Evolutionary Games
- II. Networked Evolutionary Games
- III. Large-size Networked
- IV. Exercise
- V. Appendix

Exercise 1

考虑一个有限博弈 $G=(N,S,C)$, 这里 $N=\{1,2,3\}$, $S_1=\{1,2,3\}$, $S_2=\{1,2\}$, $S_3=\{1,2,3\}$, 支付矩阵见表1.

(1). 写出该博弈的向量结构.

(2). 假设策略更新规则为 parallel MBRA, 求该策略更新规则下所确定的局势演化方程.

$C\setminus P$	111	112	113	121	122	123	211	212	213	221	222	223	311	312	313	321	322	323
c_1	1	2	-1	-2	0	1	-2	1	1	1	0	2	3	2	1	-1	2	-2
c_2	2	3	4	3	2	1	3	2	2	2	3	1	3	2	4	5	3	1
c_3	-2	-1	0	-4	-2	-3	-3	-2	0	-1	-1	0	0	-3	-3	-2	-1	-1

表1: 支付矩阵

Exercise 2

给定一个网络演化博弈, 网络图如 Figure 11 所示, 基本网络博弈为鹰鸽博弈, 支付矩阵如表 2 所示.

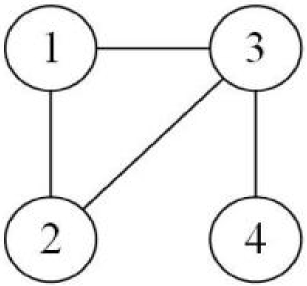


Figure 14: 网络图

$P_1 \setminus P_2$	H	D
H	(1, 1)	(4, 0)
D	(0, 4)	(2, 2)

表 2: 鹰鸽博弈支付矩阵

(1). 求出鹰鸽博弈的向量结构.

(2). 若策略更新规则为带优先级的无条件模仿, 且玩家 1 为控制, 试建立该网络演化博弈的布尔控制网络模型.

Exercise 2

(3). 试设计状态反馈控制, 使得 (2) 中建立的布尔控制网络镇定到 $x_e = \delta_8^1$, 并借此分析该网络演化博弈在 (2) 的条件下的策略一致性.

(4). 若网络图14中玩家 1 和玩家 2 之间的博弈关系每时每刻都以一定的概率出现中断, 设为 0.2. 在策略更新规则为带优先级的无条件模仿, 且玩家 1 为控制的情况下, 试建立该网络演化博弈的概率布尔控制网络模型.

Self-study

- D. Cheng, F. He, H. Qi, T. Xu, Modeling, analysis and control of networked evolutionary games, IEEE Transactions on Automatic Control, Vol. 60, No. 9, 2402-2415, 2015.
- D. Cheng, T. Xu, H. Qi. Evolutionarily stable strategy of networked evolutionary games, IEEE Transactions on Neural Networks and Learning Systems, Vol. 25, No. 7, 1335-1345, 2014.
- H. Qi, Y. Wang, T. Liu, D Cheng, Vector space structure of finite evolutionary games and its application to strategy profile convergence, Journal of Systems Science & Complexity, Vol. 29, 602-628, 2016.
- P. Guo, Y. Wang, H. Li, Algebraic formulation and strategy optimization for a class of evolutionary networked games via semi-tensor product method, Automatica, Vol. 49, 3384-3389, 2013.

- I. Evolutionary Games
- II. Networked Evolutionary Games
- III. Large-size Network
- IV. Exercise
- V. Appendix

Appendix

Definition A.1: Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote by $t := \mathrm{lcm}(n, p)$ the least common multiple of n and p. Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \quad (54)$$

Remark A.2:

- When $n = p$, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.
- When $n = rp$, denote it by $A \succ_r B$; when $rn = p$, denote it by $A \prec_r B$. These two cases are called the multi-dimensional case, which is particularly important in applications.
- The STP keeps almost all the major properties of the conventional matrix product unchanged.

Appendix

Proposition A.3:

1) (Associative Law)

$$A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C.$$

2) (Distributive Law)

$$(A + B) \ltimes C = A \ltimes C + B \ltimes C$$
$$A \ltimes (B + C) = A \ltimes B + A \ltimes C.$$

3)

$$(A \ltimes B)^T = B^T \ltimes A^T.$$

4) Assume A and B are invertible, then

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}.$$

Appendix

Proposition A.4: Let $X \in \mathbb{R}^t$ be a column vector. Then for a matrix M

$$X \ltimes M = (I_t \otimes M) \ltimes X. \tag{59}$$

Definition A.5:

$$\begin{aligned} W_{[n,m]} := \delta_{mn}[&1, m+1, 2m+1, \ldots, (n-1)m+1, \\ &2, m+2, 2m+2, \ldots, (n-1)m+2, \\ &\ldots, n, m+n, 2m+n, \ldots, mn] \\ \in \mathcal{M}_{mn \times mn}& \end{aligned} \tag{60}$$

which is called a swap matrix.

Appendix

Proposition A.6: Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two column vectors. Then

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X. \tag{61}$$

Proposition A.7:

The Khatri–Rao product of M and N, denoted by $M * N \in \mathcal{M}_{pq \times n}$, is defined column by column as follows:

$$\mathrm{Col}_i(M * N) = \mathrm{Col}_i(M) \ltimes \mathrm{Col}_i(N), \quad i = 1, \ldots, n. \tag{28}$$

Thanks！