A general survey on STP

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or with the equivalent notation $L = \delta_k \begin{bmatrix} i_1 & i_2 & \dots & i_n \end{bmatrix}$.

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- $[k]$ denotes the integer set $\{1, 2, \ldots, k\}$.
- $\mathbf{1}_k$ is the vector of size k with all unitary entries.

Definition of STP (1)

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$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1c_A}B \\ a_{21}B & a_{22}B & \dots & a_{2c_A}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{r_A1}B & a_{r_A2}B & \dots & a_{r_Ac_A}B \end{bmatrix}$$

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The size of $A \otimes B$ is $(r_A r_B) \times (c_A c_B)$.

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- The semi-tensor product $\ltimes$ between matrices $A = [a_{ij}] \in \mathbb{R}_{r_A \times c_A}$ and $B = [b_{ij}] \in \mathbb{R}_{r_B \times c_B}$ is defined as

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Example: Suppose

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}.$$

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Then $c_A = 2, r_B = 3$ and $T = 6$, so that

$$A \ltimes B = (A \otimes I_3)(B \otimes I_2) = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \\ 6 & 0 \\ 0 & 6 \\ 7 & 0 \\ 0 & 7 \end{bmatrix}$$

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$$X = 1 \longleftrightarrow x = L(X) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \delta_2^1, \quad X = 0 \longleftrightarrow x = L(X) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \delta_2^2.$$

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- It extends to a bijective correspondence between $\mathcal{B}^n$ and Δ_{2^n} through:

$$X = \begin{bmatrix} X_1 & X_2 & \dots & X_n \end{bmatrix}^\top \longleftrightarrow x = L(X_1) \ltimes L(X_2) \ltimes \dots \ltimes L(X_n).$$

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In general,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \longleftrightarrow x = L(X) = \begin{bmatrix} X_1 X_2 X_3 \\ X_1 X_2 \bar{X}_3 \\ X_1 \bar{X}_2 X_3 \\ X_1 \bar{X}_2 \bar{X}_3 \\ \bar{X}_1 X_2 X_3 \\ \bar{X}_1 X_2 \bar{X}_3 \\ \bar{X}_1 \bar{X}_2 X_3 \\ \bar{X}_1 \bar{X}_2 \bar{X}_3 \end{bmatrix}.$$

Special matrices

- Given any two vectors $X \in \Delta_n$ and $Y \in \Delta_m$, the swap matrix $W_{[n,m]}$ is the (uniquely determined) permutation matrix such that

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- Given any vector $X \in \Delta_n$, the power reducing matrix Φ_n is the (uniquely determined) matrix such that

$$X \ltimes X = \Phi_n X.$$

Boolean networks

A Boolean Network (BN) is described by the following equations

$$\begin{aligned} X(t+1) &= f(X(t)), \\ Y(t) &= h(X(t)), \qquad t \in \mathbb{Z}_+, \end{aligned} \tag{1}$$

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$$\begin{aligned} f &: \mathcal{B}^n \to \mathcal{B}^n, \\ h &: \mathcal{B}^n \to \mathcal{B}^p. \end{aligned}$$

Boolean Control Networks

A Boolean control network (BCN) is described by the following equations

$$\begin{aligned} X(t+1) &= f(X(t), U(t)), \\ Y(t) &= h(X(t)), \qquad t \in \mathbb{Z}_+, \end{aligned} \tag{2}$$

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BNs and BCNs: from logic to algebraic representations (1)

Given any logical map

$$Z = g(X),$$

If we represent the Boolean vectors X and Z by means of their “canonical equivalent", x and z, then we can always find a logical matrix M_g, called structure matrix, such that

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Consequently, every BN (1) can be described as

$$\begin{aligned} \mathbf{x}(t+1) &= L \ltimes \mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \\ \mathbf{y}(t) &= H\mathbf{x}(t) \end{aligned} \tag{3}$$

where $L \in \mathcal{L}_{2^n \times 2^n}$ and $H \in \mathcal{L}_{2^p \times 2^n}$.

BNs and BCNs: from logic to algebraic representations (2)

Similarly, every BCN (2) can be described as

$$\begin{aligned} \mathbf{x}(t+1) &= L \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \\ \mathbf{y}(t) &= H\mathbf{x}(t) \end{aligned} \tag{4}$$

where $L \in \mathcal{L}_{2^n \times 2^{(n+m)}}$ and $H \in \mathcal{L}_{2^p \times 2^n}$.

Thanks for your attention!

Questions?