An Introduction to Axiomatic STP

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Outline

I. Axiomatic Definition for STP

☞ Cross-Dimensional Objects

Vector Space:

$$\mathbb{R}^\infty := \bigcup_{n=1}^{\infty} \mathbb{R}^n.$$

Linear Operator:

$$\mathcal{M} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{M}_{m\times n}.$$

☞ Axiomatic STP

Definition 1.1

(1) Let $A,\ B \in \mathcal{M}$. A matrix-matrix (MM-) STP is a mapping $\pi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, satisfying

(i) (Consistency) When $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and $n = p$, it coincides with standard matrix product, i.e.,

$$\pi(A,B) = AB. \tag{1}$$

(ii) (Associativity)

$$\pi(A,\pi(B,C)) = \pi(\pi(A,B),C), \quad A,B,C \in \mathcal{M}. \tag{2}$$

(iii) (Distributivity) Let $A,\ B \in \mathcal{M}_{m\times n}$. Then

$$\pi(A+B,C) = \pi(A,C) + \pi(B,C). \tag{3}$$

Definition 1.1(cont'd)

(2) Let $A \in \mathcal{M}_{m\times n}$ and Let $x \in \mathbb{R}^r$. A matrix-vector (MV-) STP is a mapping $\pi : \mathcal{M} \times \mathbb{R}^\infty \to \mathbb{R}^\infty$, satisfying

- **(i)** (Consistency) when $n = r$, it coincides with standard matrix-vector product. That is,

$$\pi(A, x) = Ax. \tag{4}$$

- **(ii)** (Associativity) Assume the product of A and B exists, then

$$\pi(A, \pi(B, x)) = \pi(AB, x). \tag{5}$$

- **(iii)** (Distributivity) Let $A,\ B \in \mathcal{M}_{m\times n},\ x,\ y \in \mathbb{R}^p$. Then

$$\begin{aligned} \pi(A+B, x) &= \pi(A, x) + \pi(B, x); \\ \pi(A, x+y) &= \pi(A, x) + \pi(A, y). \end{aligned} \tag{6}$$

Definition 1.1(cont'd)

(3) Let $x \in \mathbb{R}^s$, $y \in \mathbb{R}^t$. A vector-vector (VV-) STP is a mapping $\pi : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$, satisfying

(i) when $s = t$, it coincides with standard inner product. That is

$$\pi(x, y) = \langle x, y \rangle . \tag{7}$$

(ii) (Commutativity)

$$\pi(x, y) = \pi(y, x). \tag{8}$$

$$\tag{9}$$

(iii) (Distributivity) Let $x,\ y \in \mathbb{R}^n$. Then

$$\begin{array}{l} \pi(x + y, z) = \pi(x, z) + \pi(y, z); \\ \pi(z, x + y) = \pi(z, x) + \pi(z, y). \end{array} \tag{10}$$

II. Commonly Used STP

☞ MM-STP

Definition 2.1

(MM-STP) Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, $t = \operatorname{lcm}(n, p)$.

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right). \tag{11}$$

Proposition 2.2

(i) Consistency; Associativity; Distributivity.

(ii)

$$(A \ltimes B)^T = B^T \ltimes A^T. \tag{12}$$

(iii) Assume A and B are invertible, then

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{13}$$

☞ MV-STP

Definition 2.3

(MV-STP) Let $A \in \mathcal{M}_{m\times n}$, $x \in \mathbb{R}^p$, $t = \operatorname{lcm}(n, p)$.
$\vec{\ltimes} : \mathcal{M} \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined by

$$A \,\vec{\ltimes}\, x := \left(A \otimes I_{t/n}\right) \left(x \otimes \mathbf{1}_{t/p}\right). \tag{14}$$

☞ VV-STP

Definition 2.4

(VV-STP) Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $t = \operatorname{lcm}(m, n)$.
$\vec{\cdot} : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$ is defined by

$$x \,\vec{\cdot}\, y := \left\langle x \otimes \mathbf{1}_{t/m}, y \otimes \mathbf{1}_{t/n} \right\rangle. \tag{15}$$

III. Dimension-Keeping STP

☞ DK-STP

Definition 3.1

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$, $t = \mathrm{lcm}(n,p)$. The DK-STP of A and B, denoted by $A \bar{\times} B \in \mathcal{M}_{m\times q}$, is defined as follows.

$$A \bar{\times} B := \left(A \otimes \mathbf{1}^T_{t/n}\right)\left(B \otimes \mathbf{1}_{t/p}\right). \tag{16}$$

Remark 3.2

(i) When $n = p$,
$$A \bar{\times} B = AB.$$

(ii) If $A, B \in \mathcal{M}_{m\times n}$, then $A \bar{\times} B \in \mathcal{M}_{m\times n}$.

D. Cheng, From DK-STP to Non-square General Linear Algebra and General Linear Group, (preprint: http:arxiv.org/abs/2305.19794v2), 2023.

Remark 3.2(cont'd)

(iii) It is MM-, MV-, and VV- STP.

(iv) $(\mathcal{M}_{m\times n}, +, \ltimes)$ is a ring.

Proposition 3.3

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$, $t = \operatorname{lcm}(n,p)$.

$$\begin{aligned} A \ltimes B &= A\left(I_n \otimes \mathbf{1}^T_{t/n}\right)\left(I_p \otimes \mathbf{1}_{t/p}\right)B \\ &:= A\Psi_{n\times p}B, \end{aligned} \tag{17}$$

where

$$\Psi_{n\times p} = \left(I_n \otimes \mathbf{1}^T_{t/n}\right)\left(I_p \otimes \mathbf{1}_{t/p}\right) \in \mathcal{M}_{n\times p}$$

is called the bridge matrix of dimension $n \times p$.

Definition 3.4

Assume $A \in \mathcal{M}_{m\times n}$. Consider $A : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $x \mapsto A \ltimes x$. Then $\mathbb{R}^m \subset \mathbb{R}^\infty$ is an invariant subspace. Denote by Π_A the restriction of $A|_{\mathbb{R}^m} = \Pi_A$. That is

$$A \ltimes x = \Pi_A x, \quad \forall x \in \mathbb{R}^m. \tag{18}$$

Proposition 3.5

$$\Pi_A = A \ltimes I_m = A\Psi_{n\times m}. \tag{19}$$

☞ Generalized Cayley-Hamilton Theorem

Definition 3.6

(i)

$$A^{<k>} := \underbrace{A \ltimes \cdots \ltimes A}_{k}. \tag{20}$$

(ii) Let $A \in \mathcal{M}_{m\times n}$ and $A|_{\mathbb{R}^m} = \Pi_A$. The characteristic polynomial of Π_A is called the characteristic polynomial of A.

Theorem 3.7

Let $A \in \mathcal{M}_{m\times n}$ and $A|_{\mathbb{R}^m} = \Pi_A$. Denote by $p(x) = x^m + p_{m-1}x^{m-1} + \cdots + p_0$ the characteristic polynomial of $\Pi(A)$. Then

$$A^{<m+1>} + p_{r-1}A^{<m>} + \cdots + p_0 A = 0. \tag{21}$$

Definition 3.8

Consider $\mathcal{M}_{m\times n}$, a Lie bracket over $\mathcal{M}_{m\times n}$, defined by using $\bar{\times}$, is

$$[A,B]_{\bar{\times}} := A \bar{\times} B - B \bar{\times} A, \quad A,B \in \mathcal{M}_{m\times n}. \tag{22}$$

Proposition 3.9

(i) $\mathcal{M}_{m\times n}$ with Lie bracket defined by (22) is a Lie algebra, denoted by $\mathrm{gl}(m \times n, \mathbb{F})$.

(ii) There exists the corresponding Lie group, denoted by $\mathrm{GL}(m \times n, \mathbb{F})$, which has $\mathrm{gl}(m \times n, \mathbb{F})$ as its Lie algebra.

IV. Conclusion

☞ Brief Conclusion:

(i) STP is a powerful tool in dealing with higher dimensional data.

(ii) According to the axiomatic definition, we are able to define various different STPs, which might be used to different fields.

(iii) DK-STP is theoretically interesting and has potential applications. It is worth to be investigated.

A new problem area is like a newly discovered mine. For the same effort you can pick up more nuggets.

– Y.C. Ho

Thank you!

Any Question?