From Dimensional-Free Manifold to Dimension-Varying (Control) Systems

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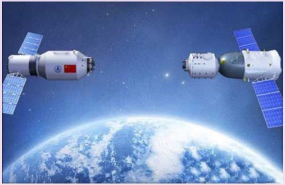
Outline

I. Introduction: Dimension-Varying Systems

☞ Background of Dimension-Varying Systems

In natura and engineering systems there are many dimension-varying systems.

Centralized DV Dystems:

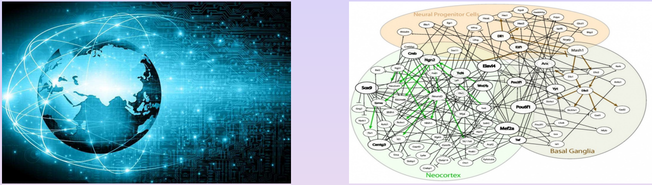


(a) Spacecraft docking

(b) Vehicle clutch system

Figure 1: Centralized DV Systems

Distributed DV Dystems:



(a) Internet (b) Genetic regulatory networks

Figure 2: Distributed DV Systems

Systems with Various Dimension Models: String

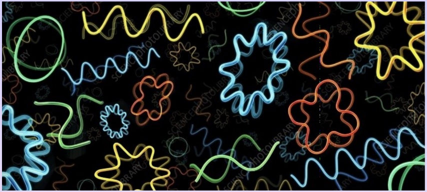


Figure 3: String in Physics

Space-time: dimension 4 (Einstein Relativity), 5 (Kalabi-Klein theory), 10 (Type 1 string), 11 (M-theory) or even 26 (Bosonic model)

M. Kaku, *Introduction to Supersting and M-Theory*, 2nd Ed., Springer-Verlag, New York, 1999.

Systems with Multiple Models: Power Generator

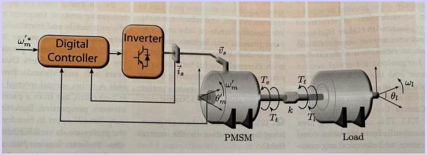


Figure 4: Stand Alone System

A single generator can be modeled as a 2, 3, or 5, 6, or even 7, dimensional dynamic system.

Models of different dimensions may be used to describe same objects!

J. Machowski, J.W. Bialek, J.R. Bumby, *Power System Dynamics and Stability*, John Wiley and Sons, Inc., Chichester, 1997.

II. Mix-Dimension Euclidean Space

☞ Natural Topology on $\mathbb{R}^\infty$

Mix-Dimension Space:

$$\begin{array}{l} \mathcal{V}_n := \mathbb{R}^n, \quad n = 1, 2, \cdots, \\ \mathcal{V}(= \mathbb{R}^\infty) := \bigcup_{n=1}^{\infty} \mathbb{R}^n. \end{array}$$

Remark 2.1

$\mathbb{R}^\infty$ has a natural topology, denoted by $\mathcal{T}_N$, which consists of

(i) each $\mathbb{R}^n$, as an Euclidian space, has its standard topology within $\mathbb{R}^n$;

(ii) over $\mathbb{R}^\infty$ each $\mathbb{R}^n$ is a clopen subset.

☞ Cross Dimensional Structure

Vector Space Structure:

Definition 2.2

Let $x \in \mathbb{R}^m \subset \mathbb{R}^\infty$, $y \in \mathbb{R}^n \subset \mathbb{R}^\infty$, $t = m \vee n$. Then

$$x \vec{\pm} y := (x \otimes \mathbf{1}_{t/m}) \pm (y \otimes \mathbf{1}_{t/n}) \in \mathbb{R}^t \subset \mathbb{R}^\infty. \tag{1}$$

$\mathbb{R}^\infty$ becomes a pseudo-vector space.

R. Abraham, J.E. Marsden, *Foundations of Mechanics*, 2nd Ed., Benjamin/Cummings Pub. London, 1978.

Topological Structure

Definition 2.3

Let $x \in \mathbb{R}^m \subset \mathbb{R}^\infty$, $y \in \mathbb{R}^n \subset \mathbb{R}^\infty$.

(i) Inner product (of x and y):

$$\begin{array}{c} \langle x, y \rangle_{\mathcal{V}} := \frac{1}{t} \left\langle (x \otimes \mathbf{1}_{t/m}), (y \otimes \mathbf{1}_{t/n}) \right\rangle, \\ x \in \mathbb{R}^m,\ y \in \mathbb{R}^n,\ t = m \vee n. \end{array} \tag{2}$$

(ii) Norm (of x):

$$\|x\|_{\mathcal{V}} := \sqrt{\langle x, x \rangle_{\mathcal{V}}}. \tag{3}$$

(iii) Distance (of x and y):

$$d_{\mathcal{V}}(x, y) := \|x \vec{-} y\|_{\mathcal{V}}. \tag{4}$$

The topology deduced by the distance $d_{\mathcal{V}}$, denoted by $\mathcal{T}_d$. Then

$$\Omega := \left(\mathbb{R}^{\infty}, \mathcal{T}_d\right). \tag{5}$$

Definition 2.4

Let $x, y \in \mathbb{R}^{\infty}$. x and y are equivalent, denoted by $x \leftrightarrow y$, if

(i) $d_{\mathcal{V}}(x, y) = 0$.

or equivalently,

(ii) There exist $\mathbf{1}_p$ and $\mathbf{1}_q$ such that

$$x \otimes \mathbf{1}_p = y \otimes \mathbf{1}_q. \tag{6}$$

Definition 2.5

(i)

$$\bar{x} := \{y \in \mathbb{R}^\infty \mid y \leftrightarrow x\}.$$

(ii)

$$\bar{x} \vec{\pm} \bar{y} := \overline{x \vec{\pm} y}. \tag{7}$$

(iii)

$$d_{\mathcal{V}}(\bar{x}, \bar{y}) := d_{\mathcal{V}}(x, y), \quad x \in \bar{x}, y \in \bar{y}, \bar{x}, \bar{y} \in \Omega. \tag{8}$$

Proposition 2.6

(i) Topologically,

$$\Omega = \mathbb{R}^\infty / \leftrightarrow . \tag{9}$$

(ii) Ω is a topological vector space.

☞ Projection From $\mathbb{R}^s$ to $\mathbb{R}^b$

Definition 2.7

Let $\xi \in \mathbb{R}^s$. The projection of ξ on $\mathbb{R}^b$, denoted by $\pi^s_b(\xi)$, is defined as

$$\pi^s_b(\xi) := argmin_{x \in \mathbb{R}_b} \|\xi \overset{\rightarrow}{-} x\|_{\mathcal{V}}, \tag{10}$$

where the norm is defined by (3).

Let $\xi \in \mathbb{R}^s$, $x \in \mathbb{R}^b$, $s \vee b = t$, and set $\alpha := t/s$, $\beta := t/b$. Then

$$\pi^s_b(\xi) = x_0 = \operatorname{argmin}_{x \in \mathbb{R}^b} \|\xi \overset{\rightarrow}{-} x\|^2_{\mathcal{V}} \in \mathbb{R}^b. \tag{11}$$

(11) yields

$$x^i_0 = \frac{t}{b}\left(\sum_{j=1}^{\beta} \eta_{(i-1)\beta+j}\right), \quad i = 1, \cdots, b. \tag{12}$$

where

$$\xi \otimes \mathbf{1}_{t/s} := (\eta_1, \eta_2, \cdots, \eta_t)^T.$$

Moreover, it is easy to verify that $\left\langle \xi \vec{-} x_0, x_0 \right\rangle_{\mathcal{V}} = 0$. Hence, we have

Proposition 2.8

Let $\xi \in \mathcal{V}_s$. The projection of ξ on $\mathcal{V}_b$, denoted by x_0, is determined by (12). Moreover, $\xi \vec{-} x$ is orthogonal to x_0. (Ref. Figure 5.)

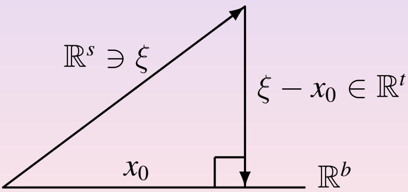


Figure 5: Cross-dimensional Projection

☞ Matrix Expression of π_b^s

Proposition 2.9

Let $s \vee b = t$, $\alpha := t/s$, and $\beta := t/b$. Then

$$\pi_b^s(\xi) = \Pi_b^s \xi, \quad \xi \in \mathcal{V}_s, \tag{13}$$

where

$$\Pi_b^s = \frac{1}{\beta} \left(I_b \otimes \mathbf{1}_\beta^T \right) \left(I_s \otimes \mathbf{1}_\alpha \right). \tag{14}$$

Proposition 2.10

1. Assume $s \geq b$, then Π_b^s is of full row rank, and hence $\Pi_b^s (\Pi_b^s)^T$ is non-singular.
2. Assume $s \leq b$, then Π_b^s is of full column rank, and hence $(\Pi_b^s)^T \Pi_b^s$ is non-singular.

☞ Projection of Linear Systems

Original System:

$$\xi(t+1) = A\xi(t), \quad \xi(t) \in \mathbb{R}^n. \tag{15}$$

Projected system:

$$x(t+1) = A_\pi x(t), \quad x(t) \in \mathbb{R}^m. \tag{16}$$

Idea Goal:

$$x(t) = \pi_m^n(\xi(t)). \tag{17}$$

Realizable Goal:

$$\min \|x(t) - \pi_m^n(\xi(t))\|. \tag{18}$$

D. Cheng, On equivalence of matrices, *Asian J. Math.*, Vol. 23, No. 2, 257-348, 2019.

☞ Least Square Approximated System

Proposition 2.11

$$A_\pi = \begin{cases} \Pi_m^n A (\Pi_m^n)^T \left(\Pi_m^n (\Pi_m^n)^T\right)^{-1} & n \geq m \\ \Pi_m^n A \left((\Pi_m^n)^T \Pi_m^n\right)^{-1} (\Pi_m^n)^T & n < m. \end{cases} \tag{19}$$

Corollary 2.12

Consider a continuous linear system

$$\dot{\xi}(t) = A\xi(t), \quad \xi(t) \in \mathbb{R}^n. \tag{20}$$

Its least square approximated system is

$$\dot{x}(t) = A_\pi x(t), \quad x(t) \in \mathbb{R}^m, \tag{21}$$

where A_π is defined by (19).

☞ For Control System

Corollary 2.13

- Consider a discrete time linear control system

$$\begin{cases} \xi(t+1) = A\xi(t) + Bu, \quad \xi(t) \in \mathbb{R}^n \\ y(t) = C\xi(t), \quad y(t) \in \mathbb{R}^p. \end{cases} \tag{22}$$

Its least square approximated linear control system is

$$\begin{cases} x(t+1) = A_\pi x(t) + \Pi_m^n Bu, \quad x(t) \in \mathbb{R}^m \\ y(t) = C_\pi x(t), \end{cases} \tag{23}$$

where A_π is defined by (19), and

$$C_\pi = \begin{cases} C(\Pi_p^n)^T \left(\Pi_p^n (\Pi_p^n)^T\right)^{-1}, \quad n \geq p \\ C\left((\Pi_p^n)^T \Pi_p^n\right)^{-1} (\Pi_p^n)^T, \quad n < p. \end{cases} \tag{24}$$

Corollary 2.13(cont'd)

- Consider a continuous time linear control system

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + Bu, & \xi(t) \in \mathbb{R}^n \\ y(t) = C\xi(t), & y(t) \in \mathbb{R}^p. \end{cases} \tag{25}$$

Its least square approximated linear control system is

$$\begin{cases} \dot{x}(t) = A_\pi x(t) + \Pi_m^n Bu, & x(t) \in \mathbb{R}^m \\ y(t) = C_\pi x(t), & y(t) \in \mathbb{R}^p, \end{cases} \tag{26}$$

where A_π is defined by (19), and C_π is defined by (24).

D. Cheng, *From Dimension-Free Matrix Theory to Cross-Dimensional Dynamic Systems*, Elsevier, London, 2019.

III. Dimension-Free Manifold

☞ Lattice Structure on $\mathbb{R}^\infty$

Definition 3.1

(i) A partial order set $L \neq \emptyset$ is a lattice, if for any two elements $x, y \in L$, there exist least common upper bound $\sup(x, y) \in L$ and greatest common lower bound $\inf(x, y) \in L$.

(ii) A subset $\emptyset \neq S \subset L$ is called a sublattice if for any two $x, y \in S$, $\sup(x, y) \in S$ and $\inf(x, y) \in S$.

(iii) A sublattice $I \subset L$ is an ideal, if for an $x \in L$ there exists $y \in I$ such that $x \prec y$, then $x \in I$.

(iv) A sublattice $F \subset L$ is a filter, if for an $x \in L$ there exists $y \in F$ such that $x \succ y$, then $x \in F$.

S. Burrus, H.P. Sankappanavar, *A Course in Universl Algebra*, Springer, New York, 1981.

☞ Hasse Diagram of Lattices

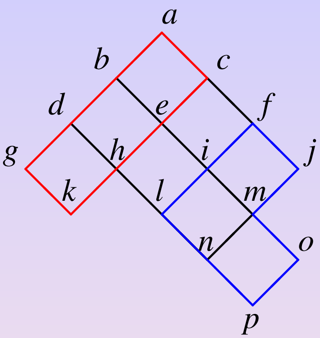


Figure 6: Hasse Diagram of a Lattice

- Filter:

$$F = \{a, b, c, d, e, g, h, k\}.$$

- Ideal:

$$I = \{f, i, j, l, m, n, o, p\}.$$

Proposition 3.2

Consider $\mathbb{R}^\infty = \bigcup_{n=1}^{\infty} \mathbb{R}^n$. Define

$$\mathbb{R}^n \prec \mathbb{R}^m \Leftrightarrow n|m.$$

Then $(\mathbb{R}^\infty, \prec)$ is a lattice, where

$$\sup(\mathbb{R}^p, \mathbb{R}^q) = \mathbb{R}^{p \vee q};$$

$$\inf(\mathbb{R}^p, \mathbb{R}^q) = \mathbb{R}^{p \wedge q}.$$

Example 3.3

(i) Define

$$\mathbb{R}^{[k,\infty)} := \{\mathbb{R}^t \mid k|t\}.$$

Then $\mathbb{R}^{[k,\infty)}$ is a filter of $\mathbb{R}^\infty$.

(ii) Define

$$\mathbb{R}^{[\cdot,k]} := \{\mathbb{R}^t \mid t|k\}.$$

Then $\mathbb{R}^{[\cdot,k]}$ is an ideal of $\mathbb{R}^\infty$.

☞ Fiber Bundle

Definition 3.4

Let T, B be two topological spaces, $\mathrm{Pr} : T \to B$ be an onto continuous mapping. (T, Pr, B) is called a fiber bundle. T: total space, B: base space, $\mathrm{Pr}^{-1}(b)$, $b \in B$, is the fiber over b.

Proposition 3.5

Let $T = (\mathbb{R}^\infty, \mathcal{T}_N)$, $B = \Omega = (\mathbb{R}^\infty, \mathcal{T}_d)$, $\mathrm{Pr} : x \to \bar{x}$. Then

$$\mathbb{R}^\infty \xrightarrow{\mathrm{Pr}} \Omega$$

is a fiber bundle.

D. Husemoller, *Fibre Budles*, 2nd Ed., Springer-Verlag, New York, 1994.

☞ Coordinate Filter (Coordinate Frame)

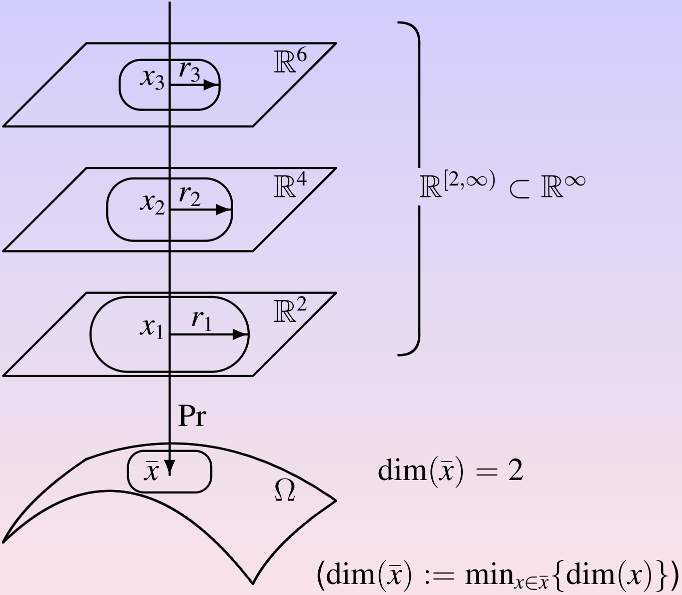


Figure 7: Bundle of coordinate filter

Differentiable Manifold Structure

- Smooth Functions ($C^r(\Omega)$):

Definition 3.6

Let $\bar{f} : \Omega \to \mathbb{R}$. Define $f_n : \mathbb{R}^n \to \mathbb{R}$ by $f_n(x) := \bar{f}(\bar{x})$, $x \in \mathbb{R}^n$. $\bar{f} \in C^r(\Omega)$ if

$$f_n \in C^r(\mathbb{R}^n), \forall n \geq 1. \tag{27}$$

Proposition 3.7

Let $f_n \in C^r(\mathbb{R}^n)$. $\bar{x} \in \Omega$ with $\dim(\bar{x}) = \dim(x_1) = m$ (where $x_1 \in \bar{x}$). Define

$$\bar{f}(\bar{x}) := f_n(\pi_n^m(x_1)), \quad \bar{x} \in \Omega. \tag{28}$$

Then $\bar{f} \in C^r(\Omega)$.

D. Cheng, Z. Ji, From dimension-free manifolds to dimension-varying control systems, *Commun. Inform. Sys.*, Vol. 23, No. 1, 85-150, 2023.

- Vector Fields ($V^r(\Omega)$):

Assume $\bar{V}$ be a set of equivalent vectors.

$$\dim(\bar{V}) := \min_{V\in\bar{V}} \dim(V). \tag{29}$$

Definition 3.8

The C^r vector field $(\bar{V})$ over Ω is a rule, which assigns to each $\bar{x}\in\Omega$ an $\bar{V}_{\bar{x}}$, satisfying

(i) $\dim(\bar{V}_{\bar{x}})$, denoted by $\mu_{\bar{x}}$, depends on $\dim(\bar{x})$ only.

(ii) There exists a $\mu<\infty$, such that

$$\mu = \max_{\bar{x}\in\Omega} \mu_{\bar{x}} < \infty. \tag{30}$$

(iii) Assume $k\succ\dim(\bar{x})\vee\mu$, then for each $x\in\mathbb{R}^k$

$$V(x)\in\bar{V}_{\bar{x}},\quad x\in\mathbb{R}^k, \tag{31}$$

is uniquely determinant. Moreover, $V(x)\in V^r(\mathbb{R}^k)$.

☞ Constructing $\bar{V}(\bar{x}) \in V^r(\Omega)$

- Tangent Space:

$$T(\Omega) = \mathbb{R}^\infty; \quad T_{\bar{x}} = \mathbb{R}^{[\dim(\bar{x}),\infty)}, \quad \bar{x} \in \Omega.$$

Algorithm 3.9

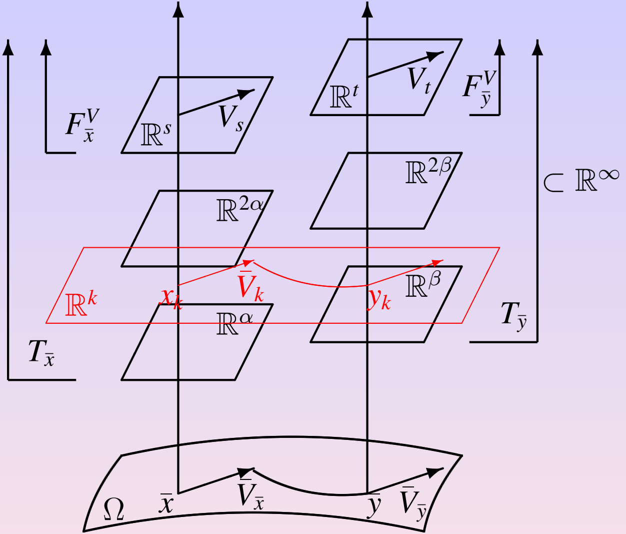
- Step 1: Find $k > 0$, the smallest dimension such that $\bar{X}$ is defined over whole $\mathbb{R}^k$ ($k \leq \mu$). Set

$$\bar{X}|_{\mathbb{R}^k} := X \in V^r(\mathbb{R}^k). \tag{32}$$

- Step 2: Extend X to $T_{\bar{x}}$. Assume $\dim(\bar{x}) = \alpha$, denote $k \vee \alpha = s$. Then $F^V_{\bar{x}} = \{\mathbb{R}^s, \mathbb{R}^{2s}, \cdots\}$ Let $\dim(x) = js$, $j = 1, 2, \cdots$. For $x \in \mathbb{R}^{js}$ define

$$V_{js} = \bar{X}(x) := \Pi^k_{js} X(\Pi^{js}_k x), \quad j = 1, 2, \cdots. \tag{33}$$

☞ Vector Field



$$\dim(\bar{x}) = \alpha,\ \dim(\bar{y}) = \beta,\ s = \alpha \vee k,\ t = \beta \vee k$$

Figure 8: Vector Field on Ω

Theorem 3.10

(i) The $\bar{X}$ generated by Algorithm 3.9 is a C^r vector field, that is, $\bar{X} \in V^r(\Omega)$.

(ii) $\bar{X} \in V^r(\Omega)$ can be generated by Algorithm 3.9.

Example 3,11

Let $X = (x_1 + x_2, x_2^2)^T \in C^\omega(\mathbb{R}^2)$. Assume $\bar{X} \in C^\omega(\Omega)$ is generated by X.

(i) Consider $\bar{y} \in \Omega$, $\dim(\bar{y}) = 3$, Denote $y_1 = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3$. Since $2 \vee 3 = 6$, $\bar{X}$ at

$$\bar{y} \bigcap \mathbb{R}^{6j} = \{y_2, y_4, y_6, \cdots\}$$

is well defined.

Example 3.11(cont'd)

Now consider y_2.

$$\bar{X}(y_2) = \Pi_6^2 X(\Pi_2^6(y_2)) = (I_2 \otimes \mathbf{1}_3) X \left(\frac{1}{3}(I_2 \otimes \mathbf{1}_3^T)(y_1 \otimes \mathbf{1}_2)\right)$$

$$= \begin{bmatrix} \frac{2}{3}(\xi_1 + \xi_2 + \xi_3) \\ \frac{2}{3}(\xi_1 + \xi_2 + \xi_3) \\ \frac{2}{3}(\xi_1 + \xi_2 + \xi_3) \\ \frac{1}{9}(\xi_2 + 2\xi_3)^2 \\ \frac{1}{9}(\xi_2 + 2\xi_3)^2 \\ \frac{1}{9}(\xi_2 + 2\xi_3)^2 \end{bmatrix}$$

Consider y_4, similar calculation shows that

$$\bar{X}(y_4) = \Pi_{12}^2 X(\Pi_2^{12}(y_4)) = \bar{X}(y_2) \otimes \mathbf{1}_2.$$

In fact, we have

$$\bar{X}(y_{2j}) = \bar{X}(y_2) \otimes \mathbf{1}_j, \quad j = 1, 2, \cdots .$$

Example 3.11(cont'd)

(ii) Consider $\bar{X}|_{\mathbb{R}^6}$:
Assume $z = (z_1, z_2, z_3, z_4, z_5, z_6)^T \in \mathbb{R}^6$. Then

$$X^6 := \bar{X}_z = \Pi_6^2 X(\Pi_2^6 z) = \begin{bmatrix} \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \end{bmatrix}. \tag{34}$$

$X^6 \in V^\omega(\mathbb{R}^6)$ is a standard vector field.

- Integral curve of $\bar{V}(\bar{x})$:

Definition 3.12

Assume $\bar{X} \in C^r(\Omega)$, $X \in C^r(R^n)$ is its generator, if $X = \bar{X}|_{\mathbb{R}^n}$. The generator of smallest dimension is called the minimum generator.

Proposition 3.13

Assume $\bar{X} \in V^r(\Omega)$.

(i) If $X \in V^r(\mathbb{R}^n)$ is its generator, then $X \otimes \mathbf{1}_s \in V^r(\mathbb{R}^{sn})$ is also its generator.

(ii) If $X \in V^r(\mathbb{R}^n)$ is its generator, $Y \in V^r(\mathbb{R}^m)$, $m < n$ is also its generator, then $m|n$, and $X = Y \otimes \mathbf{1}_{n/m}$.

(iii) Assume $\bar{X} \in V^r(\Omega)$ is dimension bounded, then it has at least one generator, and hence has a minimum generator.

Definition 3.14

Let $\bar{X} \in C^r(\Omega)$. $\bar{x}(t, \bar{x}_0)$ is called the integral curve of $\bar{X}$ with initial value $\bar{x}_0$, denoted by $\bar{x}(t, \bar{x}_0) = \Phi_t^{\bar{X}}(\bar{x}_0)$, if for each initial value $x_0 \in \bar{x}_0 \bigcap \mathbb{R}^n$, and each generator of $\bar{X}$, denoted by $X = \bar{X}|_{\mathbb{R}^n}$, the following condition holds:

$$\Phi_t^{\bar{X}}(\bar{x}_0)|_{\mathbb{R}^n} = \Phi_t^X(x_0), \quad t \geq 0. \tag{35}$$

Example 3.15

Recall Example 3.11. Let $\bar{X} \in \Omega$ be generated by

$$X = (x_1 + x_2, x_2^2)^T \in C^\omega(\mathbb{R}^2),$$

with

$$\bar{x}_0 \in \Omega \text{ and } \dim(\bar{x}_0) = 3,$$

i.e., $x_1 = (\xi_1, \xi_2, \xi_3)^T$. Find $\Phi^{\bar{X}}(\bar{x}_0)$?

Example 3.15(cont'd)

Since $2 \vee 3 = 6$, the integral curve is on $\mathbb{R}^{6k}$, $k = 1, 2, \cdots$.

First calculate the one defined on $\mathbb{R}^6$, $\mathcal{X}|_{\mathbb{R}^6} := X^6$, which is (34).

Note that $x_2^0 := \bar{x}_0 \bigcap \mathbb{R}^6$, then $x_2^0 = (\xi_1, \xi_1, \xi_2, \xi_2, \xi_3, \xi_3)^T$. Hence the integral curve is $\Phi_t^{X^6}(x_2^0)$.

It follows that

$$\Phi_t^{\bar{X}}(\bar{x}_0) = \left\{ \Phi_t^{X^6}(x_2^0) \otimes \mathbf{1}_k \mid k = 1, 2, \cdots \right\}. \tag{36}$$

☞ Other Differentiable Objects over Ω

- Distributions $D_{\bar{x}}(\Omega) \subset T_{\bar{x}}(\Omega)$:
- Covector fields $\sigma_{\bar{x}}(\Omega) \in T^*_{\bar{x}}(\Omega)$:
- Tensor fields $\mathcal{T}^r_s(\Omega)$:

$$\mathcal{T}^r_s(\Omega) : T^r(\omega) \times T^{*s}(\omega) \to \mathbb{R}.$$

- Riemannian Geometry:

All these objects can be constructed in a similar way.

D. Cheng, Z. Ji, From dimension-free manifolds to dimension-varying control systems, *Commun. Inform. Sys.*, Vol. 23, No. 1, 85-150, 2023.

IV. Dimension-Varying Dynamic (Control) Systems

☞ Projection of dynamic (control) systems

Definition 4.1

(i) Consider a dynamic system over $\mathbb{R}^p$, described as

$$\Sigma : \ \dot{x} = F(x), \quad x \in \mathbb{R}^p. \tag{37}$$

Its projection onto $\mathbb{R}^q$ is a dynamic system over $\mathbb{R}^q$, described as

$$\pi_q^p(\Sigma) : \ \dot{z} = \tilde{F}(z), \quad z \in \mathbb{R}^q, \tag{38}$$

where

$$\tilde{F}(z) = \Pi_q^p F(\Pi_p^q(z)). \tag{39}$$

Definition 4.1(cont'd)

(ii) Consider a control system

$$\Sigma^C : \dot{x} = F(x,u), \quad x \in \mathbb{R}^p, \ u \in \mathbb{R}^r. \tag{40}$$

Its projection to R^q is

$$\pi_q^p(\Sigma^C) : \dot{z} = \tilde{F}(z,u), \quad z \in \mathbb{R}^q, \ u \in \mathbb{R}^r, \tag{41}$$

where

$$\tilde{F}(z,u) = \Pi_q^p F(\Pi_p^q(z), u). \tag{42}$$

Example 4.2

Consider the following control system Σ:

$$\begin{cases} \dot{x}_1 = u_1 \sin(x_1 + x_2), \\ \dot{x}_2 = u_2 \cos(x_1 + x_2). \end{cases} \tag{43}$$

(i) Project (43) onto $\mathbb{R}^3$. It is ready to calculate that

$$\Pi_2^3 = \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad \Pi_3^2 = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Example 4.2(cont'd)

Then the projected system $\pi_3^2(\Sigma)$ is calculated as

$$\begin{cases} \dot{z}_1 = u_1 \sin(\frac{2}{3}(z_1 + z_2 + z_3)), \\ \dot{z}_2 = \frac{1}{2}\left(u_1 \sin(\frac{2}{3}(z_1 + z_2 + z_3)) + u_2 \cos(\frac{2}{3}(z_1 + z_2 + z_3))\right), \\ \dot{z}_3 = u_2 \cos(\frac{2}{3}(z_1 + z_2 + z_3)). \end{cases} \tag{44}$$

(ii) Project (43) onto $\mathbb{R}^4$. We have

$$\Pi_2^4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Pi_4^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Example 4.2(cont'd)

Then the projected system $\pi_4^2(\Sigma)$ is easily obtained as

$$\begin{cases} \dot{z}_1 = u_1 \sin(\frac{1}{2}(z_1 + z_2 + z_3 + z_4)), \\ \dot{z}_2 = u_1 \sin(\frac{1}{2}(z_1 + z_2 + z_3 + z_4)), \\ \dot{z}_3 = u_2 \cos(\frac{1}{2}(z_1 + z_2 + z_3 + z_4)), \\ \dot{z}_4 = u_2 \cos(\frac{1}{2}(z_1 + z_2 + z_3 + z_4)). \end{cases} \tag{45}$$

Proposition 4.3

Let $f(x) \in V^\infty(\mathbb{R}^p)$ and $q = kp$. Then

$$\pi_p^q \circ \pi_q^p(f(x)) = f(x). \tag{46}$$

Definition 4.4

Consider a dynamic system

$$\dot{\bar{x}} = \bar{F}(\bar{x}), \quad \bar{x} \in \Omega. \tag{47}$$

$$\dot{x} = F(x), \quad x \in \mathbb{R}^n \subset \mathbb{R}^\infty, \tag{48}$$

is called a lifting of (47), if for each $\bar{x}$ there exists $x \in \bar{x}$, such that the corresponding vector field $F(x) \in \bar{F}(\bar{x})$. Meanwhile, system (47) is called a project system of (48).

Proposition 4.5

$\bar{x}(t) = \bar{x}(t, \bar{x}_0)$ is the solution of (47), if and only if, $x(t) = x(t, x_0)$ is the solution of (48), where $x(t) \in \bar{x}(t)$, $t \in [0, \infty)$.

☞ Switching dimension-varying control systems

Assume the original control system is

$$\dot{x} = F(x,u), \quad x \in \mathbb{R}^m,\ u \in \mathbb{R}^p. \tag{49}$$

The target system is

$$\dot{z} = G(z,v), \quad z \in \mathbb{R}^n,\ v \in \mathbb{R}^q. \tag{50}$$

Our purpose is to switch system (49) to system (50) at time $t = T$. That is,

$$\bar{x}(T) = \bar{z}(T) \in \Omega. \tag{51}$$

Proposition 4.6

Assume (51) is satisfied, and assume system (49) is controllable. Then the dynamic switching from system (49) to system (50) at time $t = T$ is realizable.

Example 4.7

Consider two systems

$$\Sigma_1 : \quad \dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(0) = (0,0)^T,$$
$$\Sigma_2 : \quad \dot{z} = Pz + Qv, \quad z \in \mathbb{R}^3.$$

Design a control such that Σ_1 is switched to Σ_2 at $T = 1$. Since Σ_1 is completely controllable, and $2 \wedge 3 = 1$, so we have to design a control which can drive the system from $x(0)$ to $x(T)$ with $\dim(\bar{x}(T)) = 1$. We may choose $x(T) = (1,1)^T$ and $z(T) = (1,1,1)^T$. Then

$$\bar{x}(T) = \bar{z}(T).$$

Example 4.7(cont'd)

It is easy to calculate that the controllability Gramian matrix is

$$W_C(t) = \int_0^t e^{-A\tau} BB^T e^{-A^T\tau} d\tau = \frac{1}{6} \begin{bmatrix} 2t^3 & -3t^2 \\ -3t^2 & t \end{bmatrix}.$$

Then the control is

$$u(t) = -B^T e^{-A^T t} W_C^{-1}(T) \left(x(0) - e^{-AT} x(T) \right) = -6t.$$

Using this control, the system can be switched from Σ_1 to Σ_2 at $T = 1$.

☞ Smooth dimension-varying control systems

In this case we require a continuous $\bar{F}(\bar{x}, u)$.
Let $\bar{X}_0,\ \bar{X}_2 \in V^r(\Omega)$. Design X connecting $\bar{X}_0$ and $\bar{X}_2$ smoothly. Define

$$\bar{X} := \begin{cases} \bar{X}_0, & t \in [t_0, 0, t_1), \\ \bar{X}_1 = (1-\lambda)\bar{X}_0 + \lambda\bar{X}_2, & t \in (t_1, t_2), \\ \bar{X}_2, & t \in (t_2, \infty), \end{cases}$$

where $\lambda = \frac{t-t_1}{t_2-t_1}$.
Assume the minimum realization of $\bar{X}_0$ is $X_0 \in V^r(\mathbb{R}^p)$, the minimum realization of $\bar{X}_2$ is $X_2 \in V^r(\mathbb{R}^q)$. Then the minimum realization of $\bar{X}_1$ is $X_1 \in V^r(\mathbb{R}^{p\vee q})$. Then the integral curve of $\bar{X}$ can be lifted as shown in Fig 9.

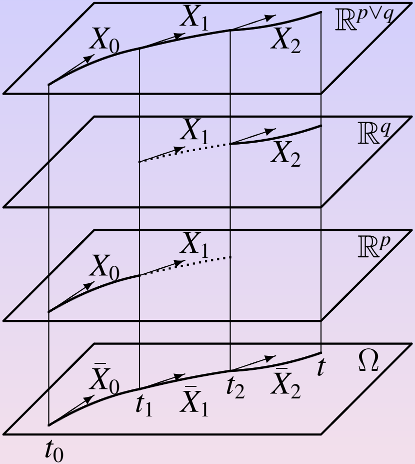


Figure 9: Lift and projection of integral curves

V. Conclusion

☞ Design Dimension-Varying (Control) Systems

(i) Construct dimension-free manifold Ω;

(ii) Build dynamic (control) system on Ω;

(iii) Design control for systems over Ω;

(iv) Lifting it to Euclidian spaces of different dimensions.

Many problems remain for further study. (Say, distributed dimension-varying systems need a different model to describe them.)

Thanks!

Any Question?