Observability Verification and Synthesis based on State-Feedback Control in Boolean Control Networks

Utilizing Observability Graph and Semitensor Product

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Workshop on STP of Matrices and Its Applications

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Content

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Monographs[a][b][c]

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[b] T. Akutsu (2018). *Algorithms for Analysis, Inference, and Control of Boolean Networks*. World Scientific.

[c] K. Zhang, L. Zhang, and L. Xie (2020). *Discrete-Time and Discrete-Space Dynamical Systems*. Communications and Control Engineering. Springer International Publishing.

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where $t \in \mathbb{Z}_+$; $x(t) \in \mathcal{B}^n, y(t) \in \mathcal{B}^q$ denote the state and output at time t, respectively; $f : \mathcal{B}^n \to \mathcal{B}^n$, $h : \mathcal{B}^n \to \mathcal{B}^q$.

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A Boolean control network

$$\begin{aligned} x(t+1) &= f(u(t), x(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{2}$$

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Observability: $(\mathcal{U}, \mathcal{Y}) \implies \mathcal{X}_0$

Observability means one can use an input sequence and the corresponding output sequence to determine the initial state of a partially-observed dynamical system, a fundamental property in computer science[a] and control science[b]

[a] E.F. Moore (1956). "Gedanken-experiments on sequential machines". In: *Automata Studies, Annals of Math. Studies* 34, pp. 129–153.

[b] R.E. Kalman (1963). "Mathematical description of linear dynamical systems". In: *Journal of the Society for Industrial and Applied Mathematics Series A Control* 1.12, pp. 152–192.

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Lay foundation for state estimation, observer design, identification, disturbance decoupling, controller synthesis, system decomposition, etc.[a]

[a] K. Zhang (2023). "A survey on observability of Boolean control networks". In: *Control Theory and Technology* 21.2, pp. 115–147.

Definition 1

A BCN (2) is called arbitrary-experiment observable if for every two different initial states, every sufficiently long input sequence is their distinguishing input sequence (DIS), i.e., for every two different initial states x_1 and x_2, under every sufficiently long input sequence U, the generated output sequences $Y(x_1, U)$ and $Y(x_2, U)$ are different.

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Necessary and sufficient conditions[a][b]

[a] E. Fornasini and M.E. Valcher (2013). "Observability, reconstructibility and state observers of Boolean control networks". In: *IEEE Transactions on Automatic Control* 58.6, pp. 1390–1401.

[b] K. Zhang and L. Zhang (2016). "Observability of Boolean control networks: A unified approach based on finite automata". In: *IEEE Transactions on Automatic Control* 61.9, pp. 2733–2738.

Content

Definition 2 (weighted pair graph[a] (later renamed observability graph))

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Consider a BCN (2). A weighted directed graph $\mathcal{G}_o = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is called its observability graph if

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Proposition 3 (K. Zhang and L. Zhang, 2014; K. Zhang and L. Zhang, 2016)

Consider a BCN (2) and its observability graph $\mathcal{G}_o = (\mathcal{V}, \mathcal{E}, \mathcal{W})$. In the diagonal subgraph $[^{\cdot}\,_{\cdot}]$ of $\mathcal{G}_o$, there is at least one cycle and every diagonal vertex will lead to a cycle. $[^{\cdot}\,_{\cdot}] \nrightarrow [_{\cdot}\,^{\cdot}]$.

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Theorem 4 (K. Zhang and L. Zhang, 2016)

A BCN (2) is not arbitrary-experiment observable (Def. 1) iff in $\mathcal{G}_o$, there is $v \in [\,_{\cdot}\,^{\cdot}\,]$ and a cycle C such that $v \longrightarrow C$.

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Example 5

Consider the following BCN

$$\begin{aligned} x_1(t+1) &= x_2(t) \wedge u(t), \\ x_2(t+1) &= \neg x_1(t) \vee u(t), \\ y(t) &= x_1(t), \end{aligned} \tag{3}$$

where $t \in \mathbb{Z}_+$; $x_1(t), x_2(t), u(t), y(t) \in \mathcal{B}$. The BCN is not arbitrary-experiment observable by its observability graph as follows and Thm. 4.

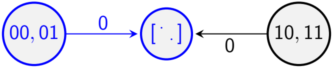

Figure 1: Observability graph of BCN (3).

Content

Semitensor product

Definition 6 (Cheng, Qi, and Z. Li, 2011)

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, and $\alpha = \text{lcm}(n, p)$ be the least common multiple of n and p. The semitensor product (STP) of A and B is defined as

$$A \ltimes B = \left(A \otimes I_{\frac{\alpha}{n}}\right) \left(B \otimes I_{\frac{\alpha}{p}}\right),$$

where $\otimes$ denotes the Kronecker product.

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- STP generalizes the conventional matrix product: If $n = p$, $A \ltimes B = AB$.
- STP preserves many good properties: associative law[a], distributive law, inverse-order laws[b], etc., so $A \ltimes B$ is usually rewritten as AB

[a] K. Zhang (2014). *On Some Control-theoretic and Dynamical Problems of Logical Dynamical Systems*. PhD Dissertation, Harbin Engineering University, China, in Chinese.

[b] D. Cheng, H. Qi, and Z. Li (2011). *Analysis and Control of Boolean Networks: A Semi-tensor Product Approach*. Springer-Verlag London.

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- $\Delta_p := \mathrm{Col}(I_p)$, $\mathcal{L}_{p\times q} = \{X \in \mathcal{B}^{p\times q} | \, \mathrm{Col}(X) \subset \mathrm{Col}(I_p)\}$.

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- $N := 2^n$, $M := 2^m$, $Q := 2^q$, $[k] := \{1, \dots, k\}$, $\delta_n^i = \mathrm{Col}_i(I_n)$.

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- $N := 2^n$, $M := 2^m$, $Q := 2^q$, $[k] := \{1, \dots, k\}$, $\delta_n^i = \mathrm{Col}_i(I_n)$.

Using STP, identify $\delta_n^i \sim \frac{n-i}{n-1}$, $i \in [n]$, BCN (2) can be equivalently transformed to[a]

$$\begin{aligned} \tilde{x}(t+1) &= L\tilde{x}(t)\tilde{u}(t) = [L_1, \dots, L_N]\tilde{x}(t)\tilde{u}(t), \\ \tilde{y}(t) &= H\tilde{x}(t), \end{aligned} \tag{4}$$

where $t \in \mathbb{Z}_+$; $\tilde{x}(t) \in \Delta_N$, $\tilde{u}(t) \in \Delta_M$, $\tilde{y}(t) \in \Delta_Q$; $L \in \mathcal{L}_{N\times NM}$ and $H \in \mathcal{L}_{Q\times N}$ are called the structure matrices, $L_i \in \mathcal{L}_{N\times M}$, $i \in [N]$.

[a] D. Cheng, H. Qi, and Z. Li (2011). *Analysis and Control of Boolean Networks: A Semi-tensor Product Approach.* Springer-Verlag London.

State-feedback controller

$$\tilde{u}(t) = G\tilde{x}(t)\tilde{v}(t) = [G_1, \dots, G_N]\tilde{x}(t)\tilde{v}(t) \tag{5}$$

with external input $\tilde{v}(t) \in \Delta_P$, $G_i \in \mathcal{L}_{M \times P}$, $i \in [N]$.

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Inserting (5) into BCN (4) yields the closed-loop state-feedback BCN

$$\begin{aligned} \tilde{x}(t+1) &= L\tilde{x}(t)G\tilde{x}(t)\tilde{v}(t), \\ \tilde{y}(t) &= H\tilde{x}(t). \end{aligned} \tag{6}$$

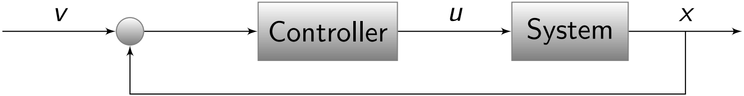

Figure 2: A closed-loop BCN based on state feedback with external input.

Proposition 7 ([a])

[a] K. Zhang (2022). "Synthesis for observability of logical control networks". In: *Automatica* 144, 110481(1–9).

Eqn (6) is equivalent to

$$\begin{aligned} \tilde{x}(t+1) &= [L_1 G_1, \dots, L_N G_N]\tilde{x}(t)\tilde{v}(t), \\ \tilde{y}(t) &= H\tilde{x}(t). \end{aligned} \tag{7}$$

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Proof.

By calculation based on STP:

$$\begin{aligned} \tilde{x}(t+1) &= [L_1, \ldots, L_N]\tilde{x}(t)\tilde{u}(t) \\ &= [L_1, \ldots, L_N]\tilde{x}(t)[G_1, \ldots, G_N]\tilde{x}(t)\tilde{v}(t) \\ &= [L_1 G_1, \ldots, L_N G_N]\tilde{x}(t)\tilde{v}(t). \end{aligned}$$

□

Content

Theorem 8 (Zhang and Johansson, 2019)

For a BCN (2)*, state feedback never enforces its controllability, but sometimes weakens its controllability; state feedback sometimes enforces its observability, sometimes weakens its observability.*

Zhang, K. and K.H. Johansson (2019). "Synthesis for controllability and observability of logical control networks". In: *2019 IEEE 58th Conference on Decision and Control (CDC)*, pp. 108–113.

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Problem 9 ([a])

[a] K. Zhang (2022). "Synthesis for observability of logical control networks". In: *Automatica* 144, 110481(1–9).

Given an unobservable BCN (2), *is there a state-feedback controller*

$$\tilde{u}(t) = G\tilde{x}(t)\tilde{v}(t)$$

as in (5) *with external input* $\tilde{v}(t)$ *such that the closed-loop BCN*

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Remark 1

Not easy to solve, because there are infinitely many state-feedback controllers with external input.

Remark 2

Problem 9 is solvable because there are finitely many state-feedback controllers $\tilde{u}(t) = G\tilde{x}(t)$ (with no external input), where $G \in \mathcal{L}_{M\times N}$, and the following result holds.

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Theorem 10 ([a])

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Consider an unobservable BCN (4). If it can be made observable by a state-feedback controller (5) with external input, then it can also be made observable by a state-feedback controller.

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Consider an unobservable BCN (4)*. If it can be made observable by a state-feedback controller* (5) *with external input, then it can also be made observable by a state-feedback controller.*

Remark 3

By Thm. 10, Problem 9 is solvable algorithmically, because there are finitely many state-feedback controllers. Problem 9 is a difficult problem because it is NP*-hard to verify observability of BCNs*[a]*.*

[a] D. Laschov, M. Margaliot, and G. Even (2013). "Observability of Boolean networks: A graph-theoretic approach". In: *Automatica* 49.8, pp. 2351–2362.

Proof of Theorem 10.

Consider an unobservable BCN Σ:

$$\begin{aligned} \tilde{x}(t+1) &= [L_1, \ldots, L_N]\tilde{x}(t)\tilde{u}(t), \\ \tilde{y}(t) &= H\tilde{x}(t), \end{aligned}$$

and a state-feedback controller $\tilde{u}(t) = G\tilde{x}(t)\tilde{v}(t)$ as in (5) with external input $\tilde{v}(t)$, such that the closed-loop BCN $\Sigma_{\tilde{u}}$:

$$\begin{aligned} \tilde{x}(t+1) &= [L_1 G_1, \ldots, L_N G_N]\tilde{x}(t)\tilde{v}(t), \\ \tilde{y}(t) &= H\tilde{x}(t) \end{aligned}$$

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is observable. By Thm. 4, in the observability graph $\tilde{\mathcal{G}}_o = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{\mathcal{W}})$ of $\Sigma_{\tilde{u}}$,

$$\text{there exists no cycle in } [\,.\,^{\cdot}\,] \quad (\circlearrowleft \notin [\,.\,^{\cdot}\,]), \tag{9a}$$

$$\text{and there exists no edge from } [\,.\,^{\cdot}\,] \text{ to } [\,^{\cdot}\,.\,] \quad ([\,.\,^{\cdot}\,] \nrightarrow [\,^{\cdot}\,.\,]). \tag{9b}$$

Proof of Theorem 10 (cont'd).

Choose a state-feedback controller

$$\hat{u}(t) = [\mathrm{Col}_i(G_1), \dots, \mathrm{Col}_i(G_N)]\,\tilde{x}(t), \tag{10}$$

by setting $\tilde{v}(t) \equiv \delta_P^i$ in $\tilde{u}(t) = G\tilde{x}(t)\tilde{v}(t)$, where $i \in [P]$, and consider the observability graph $\hat{\mathcal{G}}_o = (\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{\mathcal{W}})$ of the closed-loop BN $\Sigma_{\hat{u}}$

$$\begin{aligned} \tilde{x}(t+1) &= [L_1 \,\mathrm{Col}_i(G_1), \dots, L_N \,\mathrm{Col}_i(G_N)]\tilde{x}(t), \\ \tilde{y}(t) &= H\tilde{x}(t) \end{aligned}$$

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$$\begin{aligned} \tilde{x}(t+1) &= [L_1 \mathrm{Col}_i(G_1), ..., L_N \mathrm{Col}_i(G_N)]\tilde{x}(t), \\ \tilde{y}(t) &= H\tilde{x}(t) \end{aligned}$$

obtained by inserting (10) into (4).
$\hat{\mathcal{V}} = \tilde{\mathcal{V}}$. $\hat{\mathcal{E}} \subset \tilde{\mathcal{E}}$, because $\mathrm{Col}(L_j \mathrm{Col}_i(G_j)) \subset \mathrm{Col}(L_j G_j)$, $j \in [M]$.

Proof of Theorem 10 (cont'd).

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Theorem 11 (Zhang, 2022)

An unobservable BCN (4) can be made observable by state feedback with external input, if and only if, it can be made observable by state feedback, if and only if, there exist $i_1, \dots, i_N \in [M]$ such that the BN

$$\begin{aligned} \tilde{x}(t+1) &= [\mathrm{Col}_{i_1}(L_1), \dots, \mathrm{Col}_{i_N}(L_N)]\tilde{x}(t), \\ \tilde{y}(t) &= H\tilde{x}(t) \end{aligned} \tag{11}$$

is observable.

Zhang, K. (2022). "Synthesis for observability of logical control networks". In: *Automatica* 144, 110481(1–9).

Remark 4

Thm. 11 implies an algorithm for solving Problem 9, that is, it implies an upper bound on how many state-feedback controllers are needed to be inserted into an unobservable BCN (4) *to check if* (4) *can be made observable by state feedback with external input.*

Remark 4

Thm. 11 implies an algorithm for solving Problem 9, that is, it implies an upper bound on how many state-feedback controllers are needed to be inserted into an unobservable BCN (4) *to check if* (4) *can be made observable by state feedback with external input.*

Denote

$$\mathrm{Col}(H) = \left\{\delta_Q^{k_1}, \dots, \delta_Q^{k_\ell}\right\}, \tag{12}$$

where $\delta_Q^{k_1}, \dots, \delta_Q^{k_\ell}$ are distinct. For each $i \in [\ell]$, denote

$$\begin{aligned} S_{k_i} &:= \left\{ \delta_N^j \middle| j \in [N], H\delta_N^j = \delta_Q^{k_i} \right\}, \\ c_i &:= |S_{k_i}|, \text{ (number of states producing output } \delta_N^{k_i}) \\ S_{k_i} &=: \left\{\delta_N^{i_1}, \dots, \delta_N^{i_{c_i}}\right\}. \text{ (set of states producing output } \delta_N^{k_i}) \end{aligned} \tag{13}$$

The collection of state sets $S_{k_1}, \dots, S_{k_\ell}$ partitions Δ_N.

Theorem 12 (K. Zhang, 2022)

Consider an unobservable BCN (4). In order to verify whether (4) can be made observable by state feedback, it is sufficient to insert

$$\prod_{i=1}^{\ell} Num_i \tag{14}$$

state-feedback controllers $\tilde{u}(t) = [g_1, \dots, g_N]\tilde{x}(t)$ *with each* g_i *in* $\mathcal{L}_{M\times 1}$ *into (4) to check whether some closed-loop BN is observable, where*

$$\begin{aligned} Num_i = \Big| \Big\{ (\alpha_{i_1}, \dots, \alpha_{i_{c_i}}) \Big| \, & \alpha_{i_k} \in \mathrm{Col}(L_{i_k}), k \in [c_i], \\ & \alpha_{i_1}, \dots, \alpha_{i_{c_i}} \text{ are distinct} \Big\} \Big|, \end{aligned} \tag{15}$$

ℓ *is defined in (12),* c_i *and* $i_1, \dots, i_{c_i}$ *are defined in (13). In addition, for every two such controllers* $G_1 = [g_1^1, \dots, g_N^1]$ *and* $G_2 = [g_1^2, \dots, g_N^2]$*,*

$$\left[L_{j_1} g_{j_1}^1, \dots, L_{j_{c_j}} g_{j_{c_j}}^1 \right] \neq \left[L_{j_1} g_{j_1}^2, \dots, L_{j_{c_j}} g_{j_{c_j}}^2 \right] \text{ for some } j \in [\ell]. \tag{16}$$

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if $\alpha_{i_1}, \dots, \alpha_{i_{c_i}}$ are not distinct, then $\begin{bmatrix} & \cdot \\ \cdot & \end{bmatrix} \to \begin{bmatrix} \cdot & \\ & \cdot \end{bmatrix}$

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if (16) does not hold, then the closed-loop BNs are the same

$$\left[L_{j_1} g_{j_1}^1, \dots, L_{j_{c_j}} g_{j_{c_j}}^1\right] \neq \left[L_{j_1} g_{j_1}^2, \dots, L_{j_{c_j}} g_{j_{c_j}}^2\right] \text{ for some } j \in [\ell]. \tag{16}$$

An example I

Consider the following BCN

$$\begin{aligned}\tilde{x}(t+1) =&\delta_8[1,1,2,3,\vdots 2,3,1,4,\vdots 3,5,7,6,\vdots 6,7,8,1,\\ &2,3,7,6,\vdots 1,2,3,4,\vdots 3,4,7,8,\vdots 5,6,7,4]\tilde{x}(t)\tilde{u}(t),\\ \tilde{y}(t) =&\delta_4[1,1,1,1,1,2,2,2]\tilde{x}(t),\end{aligned} \tag{17}$$

where $t \in \mathbb{Z}_+$, $\tilde{x}(t) \in \Delta_8$, $\tilde{u}(t), \tilde{y}(t) \in \Delta_4$. The observability graph of (17) contains a path

$$\left\{\delta_8^2, \delta_8^5\right\} \xrightarrow{\delta_4^2} \left\{\delta_8^3, \delta_8^3\right\} \xrightarrow{\delta_4^1} \left\{\delta_8^3, \delta_8^3\right\},$$

hence (17) is not observable by Thm. 4.

An example II

Compute the upper bound on the number of tested state-feedback controllers.

$$\mathrm{Col}(H) = \left\{\delta_4^1, \delta_4^2\right\},$$

where denote $k_1 = 1$ and $k_2 = 2$. Then

$$S_{k_1} = \left\{\delta_8^1, \delta_8^2, \delta_8^3, \delta_8^4, \delta_8^5\right\}, \qquad c_1 = |S_{k_1}| = 5,$$
$$S_{k_2} = \left\{\delta_8^6, \delta_8^7, \delta_8^8\right\}, \qquad c_2 = |S_{k_2}| = 3,$$

$$\begin{aligned}
\mathrm{Col}(L_1) &= \{\delta_8^1, \delta_8^2, \delta_8^3\}, & \mathrm{Col}(L_2) &= \{\delta_8^1, \delta_8^2, \delta_8^3, \delta_8^4\},\\
\mathrm{Col}(L_3) &= \{\delta_8^3, \delta_8^5, \delta_8^6, \delta_8^7\}, & \mathrm{Col}(L_4) &= \{\delta_8^1, \delta_8^6, \delta_8^7, \delta_8^8\},\\
\mathrm{Col}(L_5) &= \{\delta_8^2, \delta_8^3, \delta_8^6, \delta_8^7\}, & \mathrm{Col}(L_6) &= \{\delta_8^1, \delta_8^2, \delta_8^3, \delta_8^4\},\\
\mathrm{Col}(L_7) &= \{\delta_8^3, \delta_8^4, \delta_8^7, \delta_8^8\}, & \mathrm{Col}(L_8) &= \{\delta_8^4, \delta_8^5, \delta_8^6, \delta_8^7\}.
\end{aligned}$$

$$Num_1 \cdot Num_2 = 153 \cdot 46 = 7038.$$

Observability synthesis algorithm

Algorithm 1 An observability synthesis algorithm

Input: an unobservable BCN Σ as in (4)

Output: a state-feedback controller $\mathcal{C} : \tilde{u}(t) = [g_1, \dots, g_N]\tilde{x}(t)$ with each g_i in $\mathcal{L}_{M\times 1}$ such that the closed-loop BN $\Sigma_{\mathcal{C}}$ is observable if such a $\mathcal{C}$ exists

1: initialize a state-feedback controller $\mathcal{C}_0$ and substitute it into (4)
2: **if** $\Sigma_{\mathcal{C}_0}$ is observable **then**
3:     **return** $\mathcal{C}_0$ and **stop**
4: **else**
5:     repetitively update a single column of $\mathcal{C}_0$ each time with the purpose of reducing the number of cycles in $[\,{}_{\cdot}\ {}^{\cdot}\,]$ and the number of edges $[\,{}_{\cdot}\ {}^{\cdot}\,] \to [\,{}^{\cdot}\ {}_{\cdot}\,]$ in the corresponding observability graph of $\Sigma_{\mathcal{C}_0}$ until both of them disappear, then **return** $\mathcal{C}_0$ and **stop**
6:     **if** not possible to continue with the update as in line 5 before finding a solution **then**
7:         rollback and even reinitialize $\mathcal{C}_0$, and then rerun line 5
8:     **end if**
9: **end if**

An illustrative example for Alg. 1 I

Initialize a state-feedback controller

$$\tilde{u}(t) = \delta_4[1,1,1,1,3,1,1,1]\tilde{x}(t). \tag{18}$$

Substitute (18) into (17), then get the closed-loop BN

$$\begin{aligned}\tilde{x}(t+1) =& \delta_8[{\color{blue}1},1,2,3,\vdots\,{\color{blue}2},3,1,4,\vdots\,{\color{blue}3},5,7,6,\vdots\,{\color{blue}6},7,8,1,\\ & 2,3,{\color{blue}7},6,\vdots\,{\color{blue}1},2,3,4,\vdots\,{\color{blue}3},4,7,8,\vdots\,{\color{blue}5},6,7,4]\\ & \tilde{x}(t)\delta_4[1,1,1,1,3,1,1,1]\tilde{x}(t)\\ =& \delta_8[{\color{blue}1,2,3,6,7,1,3,5}]\tilde{x}(t),\\ \tilde{y}(t) =& \delta_4[1,1,1,1,1,2,2,2]\tilde{x}(t).\end{aligned} \tag{19}$$

The observability graph of (19) as in Fig. 3 shows that (19) is not observable: $[\,_{\cdot}\ ^{\cdot}\,] \not\twoheadrightarrow [\,^{\cdot}\ _{\cdot}\,]$, but there are cycles in $[\,_{\cdot}\ ^{\cdot}\,]$.

An illustrative example for Alg. 1 II

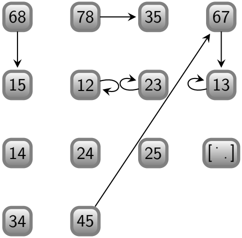


Figure 3: Observability graph of BN (19).

In order to remove the self-loop on non-diagonal vertex $\{\delta_8^1, \delta_8^2\}$, change (18) to

$$\tilde{u}(t) = \delta_4[1,4,1,1,3,1,1,1]\tilde{x}(t) \tag{20}$$

and substitute (20) into (17), then obtain the following closed-loop BN

$$\begin{aligned}\tilde{x}(t+1) &= \delta_8[1,4,3,6,7,1,3,5]\tilde{x}(t),\\ \tilde{y}(t) &= \delta_4[1,1,1,1,1,2,2,2]\tilde{x}(t).\end{aligned} \tag{21}$$

An illustrative example for Alg. 1 III

In Fig. 4, $[{}_{\cdot}{}^{\cdot}] \nrightarrow [{}^{\cdot}{}_{\cdot}]$, there is a single self-loop in $[{}_{\cdot}{}^{\cdot}]$.

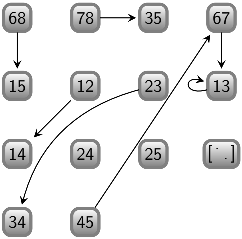


Figure 4: Observability graph of BN (21).

Furthermore change (20) to the following

$$\tilde{u}(t) = \delta_4[1, 4, 2, 1, 3, 1, 1, 1]\tilde{x}(t) \tag{22}$$

and substitute (22) into (17), then obtain the following closed-loop BN

An illustrative example for Alg. 1 IV

$$\begin{aligned} \tilde{x}(t+1) &= \delta_8[1,4,5,6,7,1,3,5]\tilde{x}(t), \\ \tilde{y}(t) &= \delta_4[1,1,1,1,1,2,2,2]\tilde{x}(t). \end{aligned} \tag{23}$$

In Fig. 5, there is no cycle in $[\,{}_{\cdot}\,{}^{\cdot}\,]$, and $[\,{}_{\cdot}\,{}^{\cdot}\,] \nrightarrow [\,{}^{\cdot}\,{}_{\cdot}\,]$, BN (23) is observable.

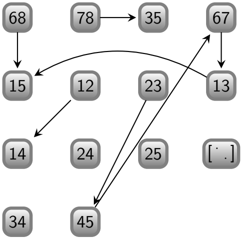


Figure 5: Observability graph of BN (23).

An illustrative example for Alg. 1 V

STEP	controller $\mathcal{C}$	closed-loop BN $\Sigma_{\mathcal{C}}$	number of cycles in $\begin{bmatrix} & \cdot \\ \cdot & \end{bmatrix}$ of the observability graph of $\Sigma_{\mathcal{C}}$
1	(18)	(19)	3
2	(20)	(21)	1
3	(22)	(23)	0

Table 1: A summary of the illustrative example for Alg. 1, where all closed-loop BNs (19), (21), and (23) satisfy (9b), i.e., $\begin{bmatrix} & \cdot \\ \cdot & \end{bmatrix} \not\rightarrow \begin{bmatrix} \cdot & \\ & \cdot \end{bmatrix}$. Finally, (23) is observable.

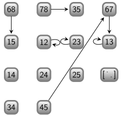
Initialize a state-feedback controller

$$\tilde{u}(t) = \delta_4[1,1,1,1,3,1,1,1]\tilde{x}(t). \tag{18}$$

Inserting (18) into (17) yields the closed-loop BN

$$\begin{aligned}\tilde{x}(t+1) =& \delta_8[1,1,2,3,\vdots 2,3,1,4,\vdots 3,5,7,6,\vdots 6,7,8,1,\\ & 2,3,7,6,\vdots 1,2,3,4,\vdots 3,4,7,8,\vdots 5,6,7,4]\\ & \tilde{x}(t)\delta_4[1,1,1,1,3,1,1,1]\tilde{x}(t)\\ =& \delta_8[1,2,3,6,7,1,3,5]\tilde{x}(t),\\ \tilde{y}(t) =& \delta_4[1,1,1,1,1,2,2,2]\tilde{x}(t).\end{aligned} \tag{19}$$

The observability graph of (19) as in Fig. 3 shows that (19) is not observable: $[\,.\,\dot{}\,] \not\twoheadrightarrow [\,\dot{}\,.\,]$, but there are cycles in $[\,.\,\dot{}\,]$.



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and substitute (20) into (17), then obtain the following closed-loop BN

$$\begin{aligned}\tilde{x}(t+1) &= \delta_8[1,4,3,6,7,1,3,5]\tilde{x}(t),\\ \tilde{y}(t) &= \delta_4[1,1,1,1,1,2,2,2]\tilde{x}(t).\end{aligned} \tag{21}$$

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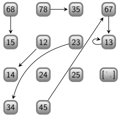

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Furthermore change (20) to the following

$$\tilde{u}(t) = \delta_4[1,4,2,1,3,1,1,1]\tilde{x}(t) \tag{22}$$

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In Fig. 5, there is no cycle in $[\,.\,\dot{}\,]$, and $[\,.\,\dot{}\,] \not\twoheadrightarrow [\,\dot{}\,.\,]$, BN (23) is observable.

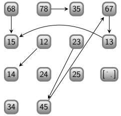

Figure 5: Observability graph of BN (23).

Summary of the observability synthesis algorithm

philosophy

Design a state-feedback controller to make the observability graph of the obtained closed-loop BCN satisfy

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- if necessary, rollback and reinitialize $\mathcal{C}_0$.

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Design a state-feedback controller to make the observability graph of the obtained closed-loop BCN satisfy

- there is no cycle in $[\,_{\cdot}\;^{\cdot}\,]$ and $[\,_{\cdot}\;^{\cdot}\,] \not\to [\,^{\cdot}\;_{\cdot}\,]$.

procedure

- Initialize a state-feedback controller $\mathcal{C}_0$,
- if necessary, change a single column of $\mathcal{C}_0$ each time to reduce the numbers of cycles in $[\,_{\cdot}\;^{\cdot}\,]$ and edges of the form $[\,_{\cdot}\;^{\cdot}\,] \to [\,^{\cdot}\;_{\cdot}\,]$,
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Summary of the observability synthesis algorithm

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- possible to find a solution but not efficient
- flexible to adjust search strategies — promising to be more efficient

Content

Definition 13

A BCN (2) is called strongly multiple-experiment observable if every initial state x_0 has a (finitely long) distinguishing input sequence (DIS) U, formally, the output sequence generated by x_0 and U is different from the output sequence generated by any x_0' different from x_0 and U.

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A BCN (2) is called multiple-experiment observable if every two different initial states x_0 and x_0' have a distinguishing input sequence (DIS) U, formally, the output sequences generated by x_0 and U and generated by x_0' and U are different.

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Definition 15

A BCN (2) is called single-experiment observable if (2) has a distinguishing input sequence (DIS) U, formally, for every two different initial states x_0 and x_0', the output sequences generated by x_0 and U and generated by x_0' and U are different.

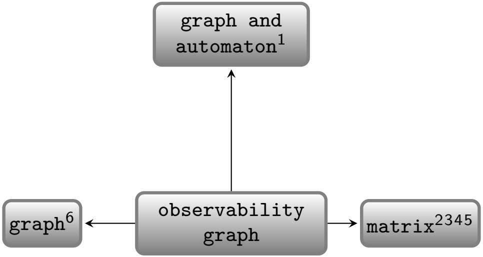

Figure 6: Representations of the fundamental idea of observability graph in different mathematical forms.

[1] K. Zhang and L. Zhang (2014). "Observability of Boolean control networks: A unified approach based on the theories of finite automata and formal languages". In: *Proceedings of the 33rd Chinese Control Conference*, pp. 6854–6861.

[2] D. Cheng, H. Qi, T. Liu, and Y. Wang (2016). "A note on observability of Boolean control networks". In: *Systems & Control Letters* 87, pp. 76–82.

[3] Y. Guo (2018). "Observability of Boolean control networks using parallel extension and set reachability". In: *IEEE Transactions on Neural Networks and Learning Systems* 29.12, pp. 6402–6408.

[4] R. Zhou, Y. Guo, and W. Gui (2019). "Set reachability and observability of probabilistic Boolean networks". In: *Automatica* 106, pp. 230–241.

[5] Y. Yu, M. Meng, J. Feng, and G. Chen (2022). "Observability criteria for Boolean networks". In: *IEEE Transactions on Automatic Control* 67.11, pp. 6248–6254.

[6] Q. Zhu, Y. Liu, J. Lu, and J. Cao (2018). "Observability of Boolean control networks". In: *Science China Information Sciences* 61.9, p. 092201.

Table 2: Complexity upper bounds for verifying observability in Boolean control networks (BCNs), where the same color represents equivalent methods.

	Def. 13	Def. 14	Def. 15	Def. 1
(Cheng and Qi, 2009) Automatica Best Paper	sufficient			
(Zhao, Qi, and Cheng, 2010)		sufficient		
(Fornasini and Valcher, 2013)				$O(2^{4n+m})$
(R. Li, M. Yang, and Chu, 2014)			$O(2^{2^{2n}+m})$	
(K. Zhang and L. Zhang, 2014) (K. Zhang and L. Zhang, 2016) (weighted pair graph (WPG), $O(2^{2n+m})$)	$O(2^{n+2^{2n}+m})$	$O(2^{4n+m})$	$O(2^{2^{2n}+m})$	$O(2^{2n+m})$
(R. Li, M. Yang, and Chu, 2015) (computational algebra, very fast in sparse BCNs)		$O(2^{2^{2n}+m})$		
(Cheng, Qi, T. Liu, and Y. Wang, 2016) observability matrix (adjacency matrix of WPG)		$O(2^{2n+m})$		
(Q. Zhu, Y. Liu, Lu, and Cao, 2018) observability graph (i.e., WPG)		$O(2^{2n+m})$		
(Y. Guo, 2018) parallel extension		$O(2^{6n+m})$	$O(2^{n2^{2n+1}+m})$	$O(2^{n2^{2n+1}+m})$

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The observability graph has been a fundamental tool in this area![a]

[a] K. Zhang (2023). "A survey on observability of Boolean control networks". In: *Control Theory and Technology* 21.2, pp. 115–147.

Thank You!

References I

Akutsu, T. (2018). *Algorithms for Analysis, Inference, and Control of Boolean Networks*. World Scientific.

Cheng, D. and H. Qi (2009). "Controllability and observability of Boolean control networks". In: *Automatica* 45.7, pp. 1659–1667.

Cheng, D., H. Qi, and Z. Li (2011). *Analysis and Control of Boolean Networks: A Semi-tensor Product Approach*. Springer-Verlag London.

Cheng, D., H. Qi, T. Liu, and Y. Wang (2016). "A note on observability of Boolean control networks". In: *Systems & Control Letters* 87, pp. 76–82.

Fornasini, E. and M.E. Valcher (2013). "Observability, reconstructibility and state observers of Boolean control networks". In: *IEEE Transactions on Automatic Control* 58.6, pp. 1390–1401.

Guo, Y. (2018). "Observability of Boolean control networks using parallel extension and set reachability". In: *IEEE Transactions on Neural Networks and Learning Systems* 29.12, pp. 6402–6408.

Ideker, T., T. Galitski, and L. Hood (2001). "A new approach to decoding life: systems biology". In: *Annual Review of Genomics and Human Genetics* 2, pp. 343–372.

Kalman, R.E. (1963). "Mathematical description of linear dynamical systems". In: *Journal of the Society for Industrial and Applied Mathematics Series A Control* 1.12, pp. 152–192.

References II

Kauffman, S.A. (1969). "Metabolic stability and epigenesis in randomly constructed genetic nets". In: *Journal of Theoretical Biology* 22.3, pp. 437–467.

Laschov, D., M. Margaliot, and G. Even (2013). "Observability of Boolean networks: A graph-theoretic approach". In: *Automatica* 49.8, pp. 2351–2362.

Li, R., M. Yang, and T. Chu (2014). "Observability conditions of Boolean control networks". In: *International Journal of Robust and Nonlinear Control* 24.17, pp. 2711–2723.

— (2015). "Controllability and observability of Boolean networks arising from biology". In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 25.2, 023104-(1–15).

Li, T., J. Feng, and B. Wang (2020). "Reconstructibility of singular Boolean control networks via automata approach". In: *Neurocomputing* 416, pp. 19–27.

Li, X., Y. Liu, J. Cao, and M. Abdel-Aty (2023). "Minimal reconstructibility of Boolean control networks". In: *IEEE Transactions on Systems, Man, and Cybernetics: Systems* 53.8, pp. 4944–4949.

Lin, L. and J. Lam (2023). "Observability categorization for Boolean control networks". In: *IEEE Transactions on Network Science and Engineering*, pp. 1–12.

Liu, Y., L. Wang, Y. Yang, and Z. Wu (2022). "Minimal observability of Boolean control networks". In: *Systems & Control Letters* 163, 105204 (1–9).

References III

Moore, E.F. (1956). “Gedanken-experiments on sequential machines”. In: *Automata Studies, Annals of Math. Studies* 34, pp. 129–153.

Wang, S. and H. Li (2021). “Graph-based function perturbation analysis for observability of multivalued logical networks”. In: *IEEE Transactions on Neural Networks and Learning Systems* 32.11, pp. 4839–4848.

Yu, Y., M. Meng, J. Feng, and G. Chen (2022). “Observability criteria for Boolean networks”. In: *IEEE Transactions on Automatic Control* 67.11, pp. 6248–6254.

Zhang, K. (2014). *On Some Control-theoretic and Dynamical Problems of Logical Dynamical Systems*. PhD Dissertation, Harbin Engineering University, China, in Chinese.

— (2022). “Synthesis for observability of logical control networks”. In: *Automatica* 144, 110481(1–9).

— (2023). “A survey on observability of Boolean control networks”. In: *Control Theory and Technology* 21.2, pp. 115–147.

Zhang, K. and L. Zhang (2014). “Observability of Boolean control networks: A unified approach based on the theories of finite automata and formal languages”. In: *Proceedings of the 33rd Chinese Control Conference*, pp. 6854–6861.

— (2016). “Observability of Boolean control networks: A unified approach based on finite automata”. In: *IEEE Transactions on Automatic Control* 61.9, pp. 2733–2738.

References IV

Zhang, K., L. Zhang, and R. Su (2016). "A weighted pair graph representation for reconstructibility of Boolean control networks". In: *SIAM Journal on Control and Optimization* 54.6, pp. 3040–3060.

Zhang, K., L. Zhang, and L. Xie (2020). *Discrete-Time and Discrete-Space Dynamical Systems*. Communications and Control Engineering. Springer International Publishing.

Zhang, X., M. Meng, Y. Wang, and D. Cheng (2021). "Observability of switched Boolean control networks via set controllability". In: *2021 60th IEEE Conference on Decision and Control (CDC)*, pp. 3912–3917.

Zhao, Y., H. Qi, and D. Cheng (2010). "Input-state incidence matrix of Boolean control networks and its applications". In: *Systems & Control Letters* 59.12, pp. 767–774.

Zhou, R., Y. Guo, and W. Gui (2019). "Set reachability and observability of probabilistic Boolean networks". In: *Automatica* 106, pp. 230–241.

Zhu, Q., Y. Liu, J. Lu, and J. Cao (2018). "Observability of Boolean control networks". In: *Science China Information Sciences* 61.9, p. 092201.

Zhu, S., J. Cao, L. Lin, L. Rutkowski, J. Lu, and G. Lu (2023). "Observability and detectability of stochastic labeled graphs". In: *IEEE Transactions on Automatic Control*, accepted.