Application of STP to Finite Games

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Outline

Modern game theory

Modern game theory began with the idea of mixed-strategy equilibria in two-person zero-sum games and its proof by John von Neumann. His paper was followed by the 1944 book Theory of Games and Economic Behavior, co-written with Oskar Morgenstern.

Introduction, Definition of Game

Definition 1. [1] A **finite game** is a triple $\mathcal{G} = (\mathcal{N},\ \mathcal{S},\ \mathcal{C})$, where

(i) $\mathcal{N} = \{1, 2, \cdots, n\}$ is the set of players;

(ii) $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ is the set of profiles, where each $\mathcal{S}_i = \{s^i_1, s^i_2, \cdots, s^i_{k_i}\}$ is the strategy set of player i;

(iii) $\mathcal{C} = \{c_1, c_2, \cdots, c_n\}$ is the set of payoff functions, where every $c_i : \mathcal{S} \to \mathbb{R}$ is the payoff function of player i.

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(iii) The payoff functions are

1, 2	rr	rp	rs	pr	pp	ps	sr	sp	ss
c_1	0	-1	1	1	0	-1	-1	1	0
c_2	0	1	-1	-1	0	1	1	-1	0

$$C_1 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

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The game of 'palm up, palm down' with 3 players:

A B C	uuu	uud	udu	udd	duu	dud	ddu	ddd
c_1	0	1	1	-2	-2	1	1	0
c_2	0	1	-2	1	1	-2	1	0
c_3	0	-2	1	1	1	1	-2	0

This matrix description of finite games based on STP was proposed in:

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Payoff matrix is defined as

$$P = \begin{bmatrix} 0 & 1 & 1 & -2 & -2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & 1 & 1 & 1 & -2 & 0 \end{bmatrix}.$$

The matrix form of payoff function:

$$c_1(x_1, x_2, x_3) = [0\ 1\ 1\ -2\ -2\ 1\ 1\ 0] x_1 x_2 x_3,$$

where

$$x_i \in \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

matrix forms of payoff matrices

In general, each payoff function can be rewritten in the matrix form based on STP as follows:

$$c_i(x_1, x_2, \cdots, x_n) = V_i^c x_1 x_2 \cdots x_n,$$

where $x_j \in \Delta_{k_j}$, $i, j = 1, 2, \ldots, n$.
The payoff matrix is an $n \times k_1 k_2 \cdots k_n$ matrix:

$$P = \begin{bmatrix} V_1^c \\ V_2^c \\ \vdots \\ V_n^c \end{bmatrix}.$$

Obviously, the **dimension** of the linear space composed of all $n \times k_1 k_2 \cdots k_n$ matrices is $n k_1 k_2 \cdots k_n$.

Definition of Potential Game

Definition.

(Monderer & Shapley, 1996) A finite game $\mathcal{G} = (\mathcal{N},\ \mathcal{S},\ \mathcal{C})$ is said to be potential if there exists a function $p : \mathcal{S} \to \mathbb{R}$, called the potential function, such that

$$c_i(x, s^{-i}) - c_i(y, s^{-i}) = p(x, s^{-i}) - p(y, , s^{-i})$$

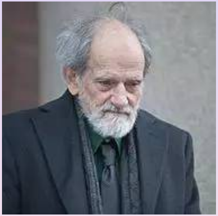
for all $x, y \in \mathcal{S}_i,\ s^{-i} \in \mathcal{S}^{-i}\ \ i = 1, 2, \cdots, n$, where $\mathcal{S}^{-i} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_n$.

$[1]$ **Monderer & Shapley,** Potential Games, **Games and Economic Behavior**, 14(1), 1996, 124-143.

potential games

Theorem

(Monderer & Shapley, 1996) Every finite potential game possesses a pure Nash equilibrium.



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(Monderer & Shapley, 1996) How can we test whether a finite game is potential?

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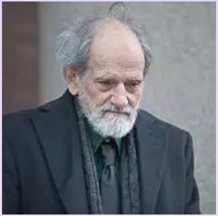

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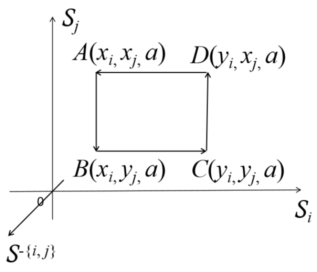
Potential Games

Theorem (Monderer & Shapley, 1996)

$\mathcal{G} = (\mathcal{N}, \mathcal{S}, \mathcal{C})$ is a potential game iff for every $i, j \in \mathcal{N}$, for every $a \in \mathcal{S}^{-\{i,j\}}$, and for every $x_i, y_i \in \mathcal{S}_i$ and $x_j, y_j \in \mathcal{S}_j$,

$$[c_j(B)-c_j(A)]+[c_i(C)-c_i(B)]+[c_j(D)-c_j(C)]+[c_i(A)-c_i(D)]=0.$$

It is called a **four-cycle equation** in (Sandholm 2010).



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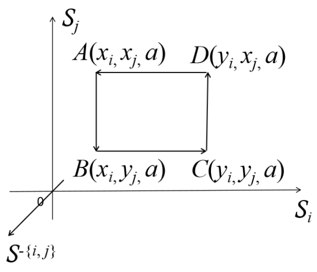

Potential Games

Question

How many equations are needed to check for a finite game with n players and k strategies for each player?

By (Monderer & Shapley, 1996), the number of equations corresponding to simple closed loops with length 4 is

$$C_n^2 k^{n-2} C_k^2 C_k^2 = \frac{n(n-1)k^n(k-1)^2}{6} = O(n^2 k^{n+2}).$$

The theoretical minimum value of the number of equations is

$$nk^n - (k^n + nk^{n-1} - 1) = (n-1)k^n - nk^{n-1} + 1 = O(nk^n).$$

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$$[c_j(B) - c_j(A)] + [c_i(C) - c_i(B)] + [c_j(D) - c_j(C)] + [c_i(A) - c_i(D)] = 0,$$

where $A = (x_i, x_j, a)$, $B = (x_i + 1, x_j, a)$, $C = (x_i + 1, x_j + 1, a)$, and $D = (x_i, x_j + 1, a)$. The number of four-cycle equations is

$$C_n^2 k^{n-2} C_k^2 C_k^2 = O(n^2 k^{n+2}).$$

By (Hino, 2011), the number of equations is

$$C_n^2 k^{n-2} (k-1)^2 = O(n^2 k^n).$$

The minimum value is $(n-1)k^n - nk^{n-1} + 1 = O(nk^n)$.

[3] **Y. Hino**, An improved algorithm for detecting potential games, **Int. J. Game Theory** (2011) 40:199-205.

potential games

U is a potential game if and only if there is a potential function V and auxiliary functions $W_p : \mathcal{S}^{-p} \to \mathbf{R}$ such that

$$U_p(s) = V(s) + W_p(s^{-p}) \ \ \forall\, s \in \mathcal{S}, \forall\, p \in \mathcal{N},$$

By using the matrix form based on STP, U is a potential game iff its payoff matrix U has the form

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} = \begin{bmatrix} V \\ V \\ \vdots \\ V \end{bmatrix} + \begin{bmatrix} W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ W_2(I_k \otimes \mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-2}}) \\ \vdots \\ W_n(I_{k^{n-1}} \otimes \mathbf{1}_k^{\mathrm{T}}) \end{bmatrix}.$$

Potential equation

A finite game U is a potential game iff there are exists row vectors V and W_i such that

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} = \begin{bmatrix} V \\ V \\ \vdots \\ V \end{bmatrix} + \begin{bmatrix} W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ W_2(I_k \otimes \mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-2}}) \\ \vdots \\ W_n(I_{k^{n-1}} \otimes \mathbf{1}_k^{\mathrm{T}}) \end{bmatrix}.$$

or

$$\begin{bmatrix} U_1 \\ U_2 - U_1 \\ \vdots \\ U_n - U_1 \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ W_2(I_k \otimes \mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-2}}) - W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ \vdots \\ W_n(I_{k^{n-1}} \otimes \mathbf{1}_k^{\mathrm{T}}) - W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \end{bmatrix}.$$

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A finite game U is a potential game iff there are exists row vectors W_i such that

$$\begin{bmatrix} U_2 - U_1 \\ \vdots \\ U_n - U_1 \end{bmatrix} = \begin{bmatrix} W_2(I_k \otimes \mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-2}}) - W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \\ \vdots \\ W_n(I_{k^{n-1}} \otimes \mathbf{1}_k^{\mathrm{T}}) - W_1(\mathbf{1}_k^{\mathrm{T}} \otimes I_{k^{n-1}}) \end{bmatrix},$$

that is,

$$\begin{bmatrix} -\mathbf{1}_k \otimes I_{k^{n-1}} & I_k \otimes \mathbf{1}_k \otimes I_{k^{n-2}} & & & \\ -\mathbf{1}_k \otimes I_{k^{n-1}} & & I_{k^2} \otimes \mathbf{1}_k \otimes I_{k^{n-3}} & & \\ \vdots & & & \ddots & \\ -\mathbf{1}_k \otimes I_{k^{n-1}} & & & & I_{k^{n-1}} \otimes \mathbf{1}_k \end{bmatrix} \xi = \begin{bmatrix} (U_2 - U_1)^{\mathrm{T}} \\ \vdots \\ (U_n - U_1)^{\mathrm{T}} \end{bmatrix}.$$

The potential equation is denoted by $\Psi\xi = b$, where Ψ is an $(n-1)k^n \times nk^{n-1}$ matrix.

Potential equation

Theorem (Cheng, Automatica, 2014)

The game $\mathcal{G} = (\mathcal{N},\ \mathcal{S},\ \mathcal{C})$ is potential if and only if the potential equation

$$\Psi\xi = b$$

has a solution ξ.

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The game $\mathcal{G} = (\mathcal{N},\ \mathcal{S},\ \mathcal{C})$ is potential if and only if

$$\mathrm{rank}[\Psi,\ b] = nk^{n-1} - 1.$$

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Potential equation

Question

What is the relationship between the four-cycle equation and the potential equation?

For the case of $n = 2$, the potential equation is

$$\left[-\mathbf{1}_{k_1} \otimes I_{k_2} \quad I_{k_1} \otimes \mathbf{1}_{k_2}\right]\xi = b.$$

Theorem

The 2-player game $\mathcal{G}$ is a potential game if and only if

$$(B_{k_1} \otimes B_{k_2})b = 0, \quad \text{i.e.,} \quad B_{k_1}(C_2 - C_1)B_{k_2}^{\mathrm{T}} = 0,$$

where $B_k = [I_{k-1}, \; -\mathbf{1}_{k-1}]$

[4] **Xinyun Liu, Jiandong Zhu**, On potential equations of finite games, **Automatica**, 68, 245-253, 2016.

Potential equation

Corollary

The 2-player game $\mathcal{G} = (C_1,\ C_2)$ is a potential game if and only if

$$r_{ij} - r_{ik_2} - r_{k_1 j} + r_{k_1 k_2} = 0 \quad (1)$$

for every $i = 1, 2, \cdots, k_1 - 1$ and $j = 1, 2, \cdots, k_2 - 1$, where $(r_{ij}) = C_2 - C_1$.

So the condition in the theorem is just a set of four-cycle equations, which has the same number of equations as the improved algorithm in (Hino, 2011).

[3] **Y. Hino**, An improved algorithm for detecting potential games, **Int. J. Game Theory** (2011) 40:199-205.

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Theorem

Consider the finite game $\mathcal{G} = (N, S, C)$. Let R_{ij} be the multi-dimensional data of the relative payoffs, i.e. $R_{ij} = V_j^c - V_i^c$. Then $\mathcal{G}$ is potential if and only if the equalities

$$(I_{k[1,t-1]} \otimes B_{k_t} \otimes I_{k[t+1,n-1]} \otimes B_{k_n})(R_{tn})^{\mathrm{T}} = 0, \qquad (2)$$

$$(I_{k[1,i-1]} \otimes B_{k_i} \otimes I_{k[i+1,j-1]} \otimes B_{k_j} \otimes I_{k[j+1,n-1]} \otimes (\delta_{k_n}^{k_n})^{\mathrm{T}})(R_{ij})^{\mathrm{T}} = 0 \qquad (3)$$

hold for all $1 \leq t \leq n-1$ and $1 \leq i < j \leq n-1$.

[4] **Xinyun Liu, Jiandong Zhu**, On potential equations of finite games, **Automatica**, 68, 245-253, 2016.

Potential equation

By (Hino, 2011), the number of equations is

$$C_n^2 k^{n-2}(k-1)^2 = O(n^2 k^n).$$

The minimum value is

$$(n-1)k^n - nk^{n-1} + 1 = O(nk^n).$$

The computation complexity of (2) is $O(nk^n)$ and that of (3) is $O(n^2k^{n-1})$.
So our algorithm has the complexity :

$$O(nk^n) + O(n^2 k^{n-1}).$$

Potential equation

Generally speaking, the computing complexity is reduced. For example, if $k = n$, the existing complexity is $O(n^{n+2})$. However, the STP method reaches the smallest possible complexity $O(n^{n+1})$. ause

The Advantages of the STP method

1. The STP method is **simpler**;
2. Not only new equivalent conditions were obtained, but also the **computing complexity was reduced**.

Vector Space Structure of Finite Games

The finite game $\mathcal{G}[n; k_1, k_2, \ldots, k_n]$ has a natural vector space structure as

$$\mathcal{G}[n; k_1, k_2, \ldots, k_n] \sim R^{nk_1k_2\cdots k_n}.$$

The decomposition of $\mathcal{G}[n; k_1, k_2, \ldots, k_n]$ was first considered by some MIT scholars [5]. They presented the following orthogonal decomposition:

$$\mathcal{G}[n; k_1, k_2, \ldots, k_n] = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{}}_{\text{Potential games}}\!\!\!\overbrace{\phantom{\mathcal{N}}\oplus \mathcal{H}}^{\text{Harmonic games}}$$

[5] **Candogan O, Menache I, Ozdaglar A**, et al. Flows and decompositions of games: Harmonic and potential games. **Mathematics of Operations Research**, 2011, 36(3): 474-503.

Vector Space Structure of Finite Games

In [5], by introducing a game graph flow, and using the Helmholtz decomposition theorem, a classical result from **algebraic topology**, the orthogonal decomposition was obtained:

$$\mathcal{G}[n;k_1,k_2,\ldots,k_n] = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{}}_{\text{Potential games}} \overbrace{\oplus\ \mathcal{H}}^{\text{Harmonic games}}$$

However, in the frame of STP method, only using **Linear Algebra** can derive the decomposition. Moreover, the bases of the subspaces can be easily obtained.

[6] **Cheng D , Liu T , Zhang K, Qi H**. On Decomposed Subspaces of Finite Games. **IEEE Transactions on Automatic Control**, 61(11), 2016, 3651-3656.

Symmetric Games

Definition (Symmetric Games [6])

A game $G \in \mathcal{G}_{[n;k]}$ is called a symmetric game if for any permutation $\sigma \in \mathbf{S}_n$

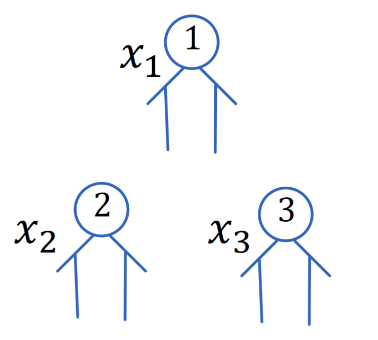
$$c_i(x_1, \cdots, x_n) = c_{\sigma(i)}(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}) \quad (4)$$

for any $i = 1, 2, \cdots, n$.

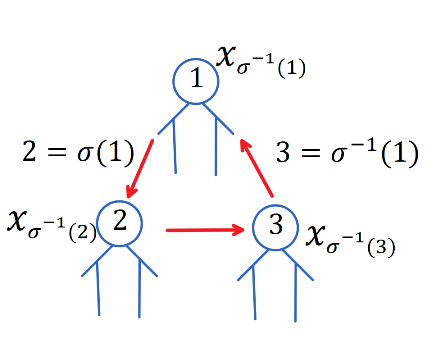
[6] **John von Neumann, Oskar Morgenstern**, Theory of games and economic behavior, Princeton University Press, 1947.

Symmetric Games

$$c_i(x_1, \cdots, x_n) = c_{\sigma(i)}(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}), \quad (i = 1, 2, \cdots, n).$$



The original strategies



The new strategies

A symmetric game means that it is fare for i and $\sigma(i)$ with respect to σ. It is just like **Equality of Opportunity**.

Symmetric Games

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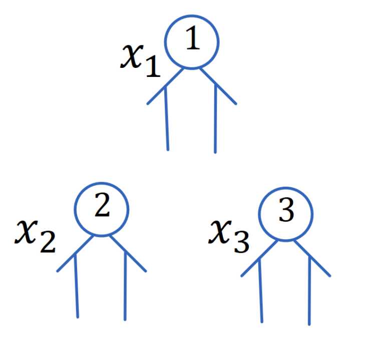

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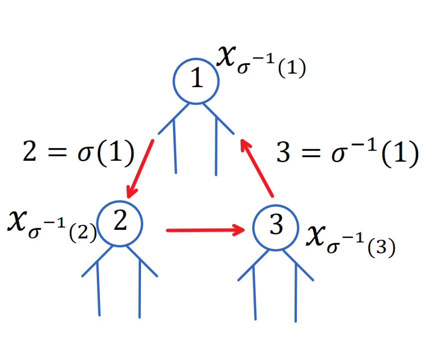

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$$c_i(x_1, \cdots, x_n) = c_{\sigma(i)}(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}) \tag{5}$$

or the matrix form based on **STP**:

$$V_i^c \ltimes_{i=1}^n x_i = V_{\sigma(i)}^c \ltimes_{i=1}^n x_{\sigma^{-1}(i)} \tag{6}$$

for any $i = 1, 2, \cdots, n$.

From the definition, it seems that we need to check (5) for every permutation σ. But actually one only needs to check every transposition $(1, j)$, because S_n is generated by $\{(1, j) | j = 2, 3, \ldots, n\}$.

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Symmetric Games

Consider

$$V_i^c \ltimes_{i=1}^n x_i = V_{\sigma(i)}^c \ltimes_{i=1}^n x_{\sigma^{-1}(i)} \tag{7}$$

for any $\sigma = (1, j) \quad (j = 2, \cdots, n)$ and $i = 1, 2, \cdots, n)$.
For the special case of $n = 4$, we can rewrite (7) as follows: when $j = 2$, we have

$$1 \overset{\sigma}{\longleftrightarrow} 2, \quad 3 \overset{\sigma}{\longrightarrow} 3, \quad 4 \overset{\sigma}{\longrightarrow} 4,$$

which correspond to the following equations, respectively:

$$\begin{aligned} V_1^c x_1 x_2 x_3 x_4 &= V_2^c x_2 x_1 x_3 x_4 = V_2^c W_{[k,k]} x_1 x_2 x_3 x_4, \\ V_3^c x_1 x_2 x_3 x_4 &= V_3^c x_2 x_1 x_3 x_4 = V_3^c W_{[k,k]} x_1 x_2 x_3 x_4, \\ V_4^c x_1 x_2 x_3 x_4 &= V_4^c x_2 x_1 x_3 x_4 = V_4^c W_{[k,k]} x_1 x_2 x_3 x_4, \end{aligned}$$

i.e.,

$$V_1^c = V_2^c W_{[k,k]}, \quad V_3^c = V_3^c W_{[k,k]}, \quad V_4^c = V_4^c W_{[k,k]}.$$

Symmetric Games

With the same argument, when $j = 3$, we have

$$V_1^c x_1 x_2 x_3 x_4 = V_3^c x_3 x_2 x_1 x_4 = V_3^c W_{[k,k]} W_{[k,k^2]} x_1 x_2 x_3 x_4,$$
$$V_2^c x_1 x_2 x_3 x_4 = V_2^c x_3 x_2 x_1 x_4 = V_2^c W_{[k,k]} W_{[k,k^2]} x_1 x_2 x_3 x_4,$$
$$V_4^c x_1 x_2 x_3 x_4 = V_4^c x_3 x_2 x_1 x_4 = V_4^c W_{[k,k]} W_{[k,k^2]} x_1 x_2 x_3 x_4,$$

which are equivalent to

$$V_1^c = V_3^c W_{[k,k]} W_{[k,k^2]}, \; V_2^c = V_2^c W_{[k,k]} W_{[k,k^2]}, \; V_4^c = V_4^c W_{[k,k]} W_{[k,k^2]}.$$

Symmetric Games

In [7], a linear representation based on STP for symmetric games is given, and necessary and sufficient conditions for symmetric games are obtained. In [8], the following result is obtained:

Theorem 20. $G \in \mathcal{G}_{[n;\kappa]}$ *is a symmetric game, if and only if,*

(i)

$$V_1^c \left[I_\kappa \otimes \left(W_{[\kappa^{s-2},\kappa]} W_{[\kappa,\kappa^{s-1}]} \right) - I_{\kappa^{s+1}} \right] = 0,$$
$$s = 2, 3, \ldots, n-1.$$

(ii)

$$V_i^c = V_1^c W_{[\kappa^{i-1},\kappa]}, \quad i = 2, 3, \ldots, n.$$

[7] **Daizhan Cheng, Ting Liu**. Linear representation of symmetric games. **IET Control Theory and Applications**, 11(18):3278–3287, 2017.

[8] **Daizhan Cheng, Ting Liu**. From Boolean game to potential game. **Automatica**, 96:51–60, 2018.

Symmetric Games

Lemma

The set of all the adjacent transpositions $(r, r+1)$, $1 \leq r \leq n-1$ is a generator of the symmetric group $\mathbf{S}_n$.

Lemma

A game $G \in \mathcal{G}_{[n;k]}$ is a symmetric game if and only if

$$c_i(x_1, \cdots, x_n) = c_{\mu_r(i)}(x_{\mu_r(1)}, \cdots, x_{\mu_r(n)}) \tag{8}$$

for any **adjacent transposition** *$\mu_r = (r, r+1)$ and i.*

[9] **Lei Wang, Xinyun Liu, Ting Li, Jiandong Zhu**, The minimum number of discriminant equations for a symmetric game, **IET Control Theory and Applications**, 16(17), 2022, 1782-1791.

Symmetric Games

In [9], we have derived a simpler equivalent condition:

Theorem

Consider $G \in \mathcal{G}_{[4;k]}$. G is a symmetric game if and only if

$$\begin{bmatrix} I_{k^4} & -T_{\mu_1} & 0 & 0 \\ 0 & I_{k^4} & -T_{\mu_2} & 0 \\ 0 & 0 & I_{k^4} & -T_{\mu_3} \\ 0 & 0 & 0 & I_{k^4}-T_{\mu_1} \\ 0 & 0 & 0 & I_{k^4}-T_{\mu_2} \end{bmatrix} (V_G)^{\mathrm{T}} = 0, \tag{9}$$

where

$$T_{\mu_1} = W_{[k]} \otimes I_{k^2}, \; T_{\mu_2} = I_k \otimes W_{[k]} \otimes I_k, \; T_{\mu_3} = I_{k^2} \otimes W_{[k]}, \tag{10}$$

and $V_G = [V_1^c \; V_2^c \; V_3^c \; V_4^c]$.

Symmetric Games

Theorem

Consider $G \in \mathcal{G}_{[4;k]}$. G is a symmetric game if and only if (9) holds, where

$$T_{\mu_1} = W_{[k]} \otimes I_{k^2}, \ T_{\mu_2} = I_k \otimes W_{[k]} \otimes I_k, \ T_{\mu_3} = I_{k^2} \otimes W_{[k]}, \tag{11}$$

and $V_G = [V_1^c \ V_2^c \ V_3^c \ V_4^c]$.

Compared with the existing result with

$$V_1^c = V_3^c W_{[k,k]} W_{[k,k^2]}, \ V_2^c = V_2^c W_{[k,k]} W_{[k,k^2]}, \ V_4^c = V_4^c W_{[k,k]} W_{[k,k^2]},$$

our expression is simpler. It is convenient for further operations.

Symmetric Games

For example, we can get the minimum number of equations.

Theorem

The linear equations with the minimum number to test symmetric games in $\mathcal{G}_{[4;k]}$ are

$$\begin{bmatrix} I_{k^4} & -T_{\mu_1} & 0 & 0 \\ 0 & I_{k^4} & -T_{\mu_2} & 0 \\ 0 & 0 & I_{k^4} & -T_{\mu_3} \\ 0 & 0 & 0 & M_{\mathcal{X}_3^\perp}^{\mathrm{T}} \otimes I_k \end{bmatrix} (V_G)^{\mathrm{T}} = 0, \qquad (12)$$

where $M_{\mathcal{X}_3^\perp}$ has the concrete expression.

Some other finite games

One can use STP method to investigate other finite games.

[10] **Yaqi Hao, Daizhan Cheng**. On skew-symmetric games. Journal of The Franklin Institute-engineering and Applied Mathematics, 355(6):3196-3220, 2018.

[11] **Changxi Li, Fenghua He, Ting Liu, and Daizhan Cheng**. Symmetry-based decomposition of finite games. Science China Information Sciences, 62(1):1 – 13, 2019.

[12] **Xiao Zhang, Yaqi Hao, and Daizhan Cheng**. Incomplete-profile potential games. Journal of the Franklin Institute, 355(2):862 – 877, 2018.

[13] **Yuhu Wu, Shuting Le, Kuize Zhang, Xi-Ming Sun**, Agent Transformation of Bayesian Games, **IEEE Transactions on Automatic Control**, 67(11), 2022, 5793-5808

Some other finite games

[14] **Wang, Yuanhua., Cheng, Daizhan and Liu, Xiyu**, Matrix expression of Shapley values and its application to distributed resource allocation. **Science China: Information Science**, 62, 22201 (2019).

[15] **Daizhan Cheng, Tingting Xu**, Application of STP to cooperative games, 2013, 10th IEEE International Conference on Control and Automation (ICCA), Hangzhou, China, 2013, pp. 1680-1685.

[16] **Daizhan Cheng, Tingting Xu, Hongsheng Qi**, Evolutionarily Stable Strategy of Networked Evolutionary Games, IEEE Transactions on Neural Networks and Learning Systems, 25(7), 1335-1345, 2014.

Conclusions

1. The STP method for finite game is simpler than the existing method. Some new conditions such as the potential equation are proposed based on STP.

2. By using the STP method, the computing complexity can be improved.

3. Many other finite games can be considered by using STP.

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Many Thanks for Your Attention!